PERFORMANCE OF LIMITERS IN MODAL DISCONTINUOUS GALERKIN METHODS FOR 1-D EULER EQUATIONS

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1-D 오일러 방정식에 관한 Modal 불연속 갤러킨 기법에서의 Limiter 성능 비교

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Considerable efforts are required to develop a monotone, robust and stable high-order numerical scheme for solving the hyperbolic system. The discontinuous Galerkin(DG) method is a natural choice, but elimination of the spurious oscillations from the high-order solutions demands a new development of proper limiters for the DG method. There are several available limiters for controlling or removing unphysical oscillations from the high-order approximate solution; however, very few studies were directed to analyze the exact role of the limiters in the hyperbolic systems. In this study, the performance of the several well-known limiters is examined by comparing the high-order(p^1, p^2, and p^3) approximate solutions with the exact solutions. It is shown that the accuracy of the limiter is in general problem-dependent, although the Hermite WENO limiter and maximum principle limiter perform better than the TVD and generalized moment limiters for most of the test cases. It is also shown that application of the troubled cell indicators may improve the accuracy of the limiters under some specific conditions.

Key Words : Discontinuous Galerkin(DG) method, hyperbolic system, 1-D Euler equations, limiters

1. Introduction

As solving real fluid dynamic problems introduce new challenges like modeling very large scale complex systems, there is increasing demand for advanced(more accurate and powerful) computational algorithms. Discontinuous Galerkin (DG) method has recently found its way into the mainstream of CFD as an alternative to finite volume and finite element methods. DG method is not only very compact, but also conservative, stable, and robust with strong mathematical supports. In particular, it is suitable for unstructured grids, parallelization, and *hp*-adaptivity[1-6].

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DG method requires very simple treatment at the boundaries to achieve the uniform high-order of accuracy throughout the domain, at least for smooth problems; therefore, it can be easily appled to solving complex flows and geometries. DG method is basically built upon two advantageous features commonly associated with finite element(FEM) and finite volume methods(FVM). Similar to continuous FEM method, it obtains the solution in an element by expanding a high-order polynomial. It also considers the physics of the wave propagation to estimate a unique solution at the interface of the elements, similar to the finite volume and finite difference methods[7]. These features make DG very powerful for solving hyperbolic systems; however, there are certain challenging issues in the development of the DG methods that require further investigation.

DG method, similar to other high-order methods, suffers from the existence of the spurious oscillations near the discontinuities. In general, there are three approaches to eliminate spurious oscillations from a numerical solution: artificial viscosity, digital filter, and limiter. Among these, the approach based on limiters may be preferable, since they can be used not only to control wiggles but also to enforce the nonlinear stability to the numerical scheme[8]. The choice of limiter functions can be categorized into three different types: 1) slope limiter, 2) spectral limiter, and 3) non-oscillatory limiter.

Slope limiters were initially developed for finite volume method. They can be categorized into several branches according to the limiting stencil, limiting process, and the shape of the limiting function. Among available slope limiters, total variation diminishing(TVD), total variation bounded(TVB), local extremum diminishing(LED), essentially local extremum diminishing(ELED), monotonicity preserving(MP), and multi-dimensional limiters[9,10] are often employed in the high-order numerical methods.

The spectral limiters have originally been developed for high-order spectral methods. In fact, there are few spectral limiters for DG methods such as high-order sub-cell limiter, moment limiter, and modified moment limiter. In these limiters, the moments of the solution are limited in a hierarchical manner. The highest-order moment of the solution is first corrected. The lower-order moments are then considered for further correction, if the highest-order moment has been changed in previous step. These limiters are usually used with a trouble cell indicator to minimize the numerical diffusion enforced to the solution due to mean of the limiters in smooth regions.

Non-oscillatory reconstruction schemes were introduced first by Harten[11] and had been developed further by several researchers[7,12]. Although the monotonicity of the solution is not guaranteed, non-oscillatory schemes can be used as the high-order schemes. It is also possible to use them as a high-order limiter in DG framework owing to their capability in preservation of the high-order solution in stiff regions. In general, non-oscillatory limiters consider an/several oscillatory high-order solution(s) to reconstruct a non-oscillatory high-order solution in the trouble element. However, the use of traditional non-oscillatory limiters are restricted to pure scientific researches due to their disadvantages such as cumbersome implementation in unstructured grids and use of very large stencils during reconstruction. Recently, a new compact and simple Hermite WENO limiter[13] has been proposed that can overcome the shortcomings of the traditional WENO limiters for structured and unstructured meshes.

In this study, we aim to examine the level of accuracy of various limiters for the one-dimensional hyperbolic systems, in particular, gas dynamic Euler equations. A high-order explicit DG method is first developed with special attention to the application of the limiters for solving the linear and nonlinear advection equation and Euler equations. Modal DG method is employed, since it has been widely used in DG community due to its advantages[14,15] over nodal DG method. Several benchmark problems with available exact analytical solutions were then considered for verification. Finally, the accuracy of the limiters in capturing discontinuities was examined in detail by comparing the profile of the numerical solutions with the exact solutions.

2. Governing equations

Euler equations describe the pure convection of the flow quantities in an inviscid fluid. In many applications, for instance, high Reynolds number flows, where a boundary layer is very thin compared to the dimension of the body, it is a valid assumption to neglect the viscous effects and employ the Euler model. Application of Euler equations enables to observe important phenomena such as compressive shock waves, expansive waves, contact discontinuity, and vortices around sharp edges. Moreover, Euler equations can serve as the basis for the development of numerical discretization methods and boundary conditions. Nonetheless, numerical solutions of Euler equations may be considered acceptable only if the equations are solved in a conservative way. The dimensionless form of the one-dimensional hyperbolic Euler system can be expressed in conservative form as follows,

$$\int_{\Omega} \frac{\partial \vec{U}}{\partial t} dV + \oint_{\partial\Omega} \vec{F}_{inv}(\vec{U}).\vec{n} dS = 0 \tag{1}$$

where conservative variables $\vec{U}$ and inviscid flux $\vec{F}_{inv}$ vector are defined as

$$\vec{U} = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}, \qquad \vec{F}_{inv} = \begin{pmatrix} \rho u \\ \rho u u + \frac{1}{N_\delta Re} p \\ (\rho E + \frac{1}{N_\delta Re} p) u \end{pmatrix}, \tag{2}$$

where N_δ and Re are a composite number[16] and the Reynolds number, respectively; ρ is the mass density; u is the velocity in the x-direction; p is the scalar pressure, and E is the total energy density. The system (1), after applying the method of line(MOL) to decouple the time and spatial coordinates, can be transformed into a decoupled system in time and space. In the present work, a modal DG method is used for discretizing the spatial coordinate and a 3rd-order strong stability preserving(SSP) method is used for time-marching.

3. Numerical method: discontinuous Galerkin

A class of numerical schemes, particularly efficient for the high-order approximation of the CFD applications, are the spectral *hp* methods[17]. According to the terminology defined in [17], spectral methods are those in which the numerical solution is fitted by series of modal functions and the accuracy of the approximate solution improves with increasing the number of modal functions. Based on the definition of the polynomial space (ansatz) function, spectral *hp* methods can be classified into several categories. The most popular high-order spectral methods are discontinuous Galerkin(DG), spectral difference(SD), spectral volume(SV), and correction procedure via reconstruction(CPR) methods.

The choice of the polynomial space function and test function are the main features that distinguish high-order methods from finite difference and finite volume methods. In classical finite element methods(i.e., continuous finite element method), the global solution is discretized using the finite dimensional functions which are locally continuous in character with finite regularities that attempt to represent the shape of the true solution. Therefore, they may not always produce the true solution at some flow conditions like high-speed conditions[18]. In contrast, high-order spectral methods consider globally smooth functions as the test functions and allow more freedom to define polynomial space function which provides enough capability to study the high-speed problems.

In continuous Galerkin(CG) method, a Hilbert space $H^1(\Omega)$ is used to approximate the finite polynomial space function. It means that, if function $f(x)$ belongs to Hilbert space $H^1(\Omega)$, it has to be continuous across the elements and satisfy $\int_\Omega (f(x)^2 + \nabla f(x).\nabla f(x))d\Omega \leq \infty$. On the other hand, discontinuous Galerkin(DG) method uses a least-square space function $L^2(\Omega)$ for approximating the polynomial space function, therefore, the space function needs to be continuous inside the element space but not over the elemental interfaces, as shown in fig. 1. In DG method, degree of freedoms(DOFs) are overlapped on the elemental edges and vertices. Therefore, the computational cost of DG method for inverting the mass matrix is higher than that of CG method. Nonetheless, DG may not always be more expensive than CG method, since the application of additional DOFs in DG method yields more accurate solutions than CG method on the same mesh. In addition, owing to the discontinuous polynomial space function and usage of an upwind monotone numerical flux function at the interface of the elements[3], DG method is often considered the best choice for numerically solving convective-dominated problems.

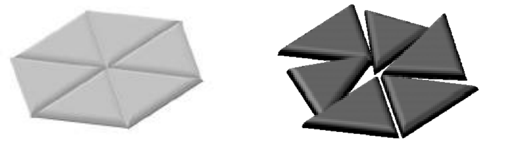


Fig. 1 Discontinuous Galerkin solution versus continuous Galerkin solution

DG methods are categorized into three types depending on the type of basis function used in the polynomial expansion series: 1) modal, 2) nodal, and 3) hybrid DG methods. Modal method may be preferred as the base of the next generation of the numerical models in the computational fluid dynamics. Modal DG method combines high-order time integration scheme, for example, the Runge-Kutta method, with the traditional DG method for solving hyperbolic systems. This method was initially developed by incorporating the numerical flux function with TVB slope limiters into the original DG framework[3]. In this study a modal Runge-Kutta DG (RKDG) method is developed to study the performance of various limiters in the DG method for hyperbolic systems, in particular, gas dynamic Euler equations.

In order to discretize Eq. (1), the exact solution of $\vec{U}$ is approximated by series of Legendre polynomials of degree of p, and the computational domain is tessellated by the bounded non-overlapping control volumes $T_h = \{\Omega_e\}$,

$$U_h(x,t) = \sum_{i=0}^{p+1} \hat{u}_h^i(t)\varphi^i(x) \tag{3}$$

where $\hat{u}_h^i(t)$ is the i^{th} local degree of the freedoms of the conservative variable approximated solution. $\varphi^i(x)$ is the i^{th} modal Legendre basis function. Considering the method of weighted residual and Galerkin method, the governing Eq. (1) are multiplied with the test function which is taken to be equal to the basis function and then integrated by parts over an element. It results in the formulation of the system for U_h as follows

$$\int_\Omega \frac{\partial \overrightarrow{U}}{\partial t}\varphi(x)dV - \int_\Omega \nabla\varphi \overrightarrow{F}_{inv}(\overrightarrow{U})dV + \oint_{\partial\Omega} \varphi \overrightarrow{F}_{inv}(\overrightarrow{U}).\overrightarrow{n}dS = 0 \tag{4}$$

where **n** is the outward unit normal vector. V and S represent the volume and boundary of the element Ω, respectively.

The weak formulation of the modal DG method can be obtained by considering the sequence of mathematical manipulations and replacing the physical flux functions with numerical functions at interfaces as

$$\sum_{e\in T_h}\int_\Omega \varphi_{h_i^e}\frac{\partial u_h^i}{\partial t}dV - \sum_{e\in T_h}\int_\Omega \frac{\partial \varphi_{h_i^e}}{\partial t}F_{inv}(\overrightarrow{U_{h^e}})dV + \sum_{e\in T_h}\oint_{\partial\Omega}\varphi_{h_i^e}\hat{f}(\overrightarrow{U_{h^e}^-},\overrightarrow{U_{h^e}^+})n_x dS = 0. \tag{5}$$

where $u_{h_i^e}$ is the i^{th}-component of the conservative variable vector $\overrightarrow{U}$, n_x is the outward unit normal vector component in *x*-direction, and $\varphi_{h_i^e}$ denotes the i^{th} test function which is set to be similar to the Legendre basis function for modal Galerkin method.

In order to solve the system (5) numerically, a series of the numerical tasks, such as determination of the volume and surface integrals in the standard element, evaluation of the numerical flux functions, calculation of the solution projection in every local element, and estimation of the local degree of freedoms for each of the independent variables, are needed.

In the present work, linear mapping is considered for transforming real element to standard element. Gauss-Legendre quadrature rule has been implemented for both the volume and the boundary integrations. The Rusanov(called as local Lax-Friedrichs) flux is applied for discretization of the inviscid flux functions at interfaces.

4. Limiters

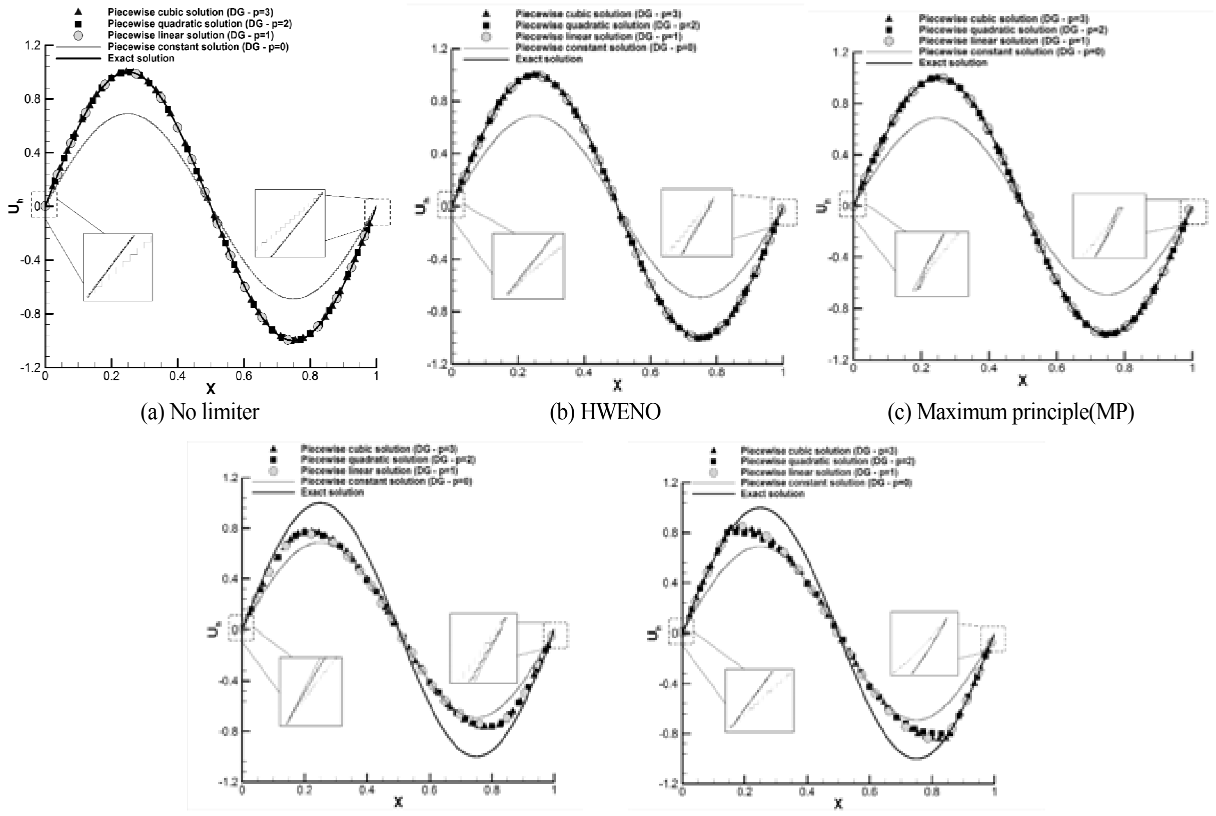
DG method, similar to other high-order methods affected by an inherent conflict between monotonicity -preserving and high-order accuracy, suffers from the existence of the spurious oscillations near the discontinuities like shock front in gas dynamics. To control these oscillations in acceptable level in DG framework, a limiter was first implemented by Chavent and Cockburn in 1989[19]. They developed Van Leer's TVD-type limiter for improving the stability of the DG method. They showed that the slope limiter must be applied not only in the stiff regions but also in some part of the smooth regions that are affected by wiggles. Goodman and LeVeque[20] also proved that any multi-dimensional TVD scheme is at most first order of accuracy in stiff flow regions. As it is not possible to develope high-order schemes with the TVD property in which high-order solutions are maintained uniformly throughout of the domain, developing more accurate new limiters was considered an important issue in the DG community.

Cockburn et al.[3] developed a TVB slope limiter to improve the performance of TVD limiters by relaxing the monotonicity constraints. The TVB slope limiter employs the modified Minmod function instead of the TVD-Minmod function in order to maintain the formal accuracy of the scheme at the extremum. Although it yields moderately improved results in various applications, the definition of the adjusting user input parameter is still questionable. Moreover, it is not possible to extend this limiter to multi-dimensional applications[5]. For this reason, several alternative limiters including generalized moment(GM) limiter[21] and maximum principle(MP) limiter[22] were proposed to preserve monotonicity near the discontinuous regions. Among these limiters, some are well-received in the DG community.

In this study, TVD[3,19,20] limiter, GM limiter[21], MP limiter[22], and Hermite WENO(HWENO) limiter[7] in conjunction with KXRCF(Krivodonova-Xin-Remacle -Chevaugeon-Flaherty)[23] trouble cell indicator and positivity-preserving feature[22] are considered. Their performance in capturing discontinuities is analyzed by comparing limited and unlimited approximate DG solutions with the exact solutions.

5. Results and discussion

We consider exact solutions of the scalar hyperbolic equation, inviscid Burger equation, and inviscid Euler equations, which are the well-known benchmark problems to examine the level of accuracy of the limiters. The



(a) No limiter (b) HWENO (c) Maximum principle(MP)

(d) Generalized moment(GM) (e) TVD

Fig. 2 The smooth solutions of the linear advection problem at $t = 4$ seconds

following scalar hyperbolic equation with the initial sinusoidal distribution of the amplitude is considered as the first benchmark problem,

$$\begin{aligned} &\partial_t u + \partial_x u = 0, \ \ -1 \le x \le 1, t \ge 0, \\ &u(x,0) = \sin(2\pi x), \\ &u(x_L,t) = u(x_R,t). \end{aligned} \tag{6}$$

The periodic boundary condition is applied to the both sides of the computational domain, and the length of the computational domain is chosen equal to one wavelength. The exact solution of this problem is very smooth and, therefore, employing limiters are not technically necessary. Nonetheless, this problem can be considered an important case to measure the loss of accuracy inflicted by the limiters.

Fig. 2 shows the distribution of the DG solutions throughout the 200 elements at $t = 4$ seconds. It can be observed that the first-order approximation of the solution is very diffusive even for this smooth problem. The sinusoidal wave is significantly damped with time in case of piecewise constant approximation; however, higher-order unlimited approximations yield very accurate results. It is obvious that the accuracy of the unlimited high-order solutions is not compromised and the high-order solutions remain very smooth.

Fig. 2 also shows that there is no spurious oscillations and wiggles within the distributed solution. This is not surprising because there are no stiff regions in the computational domain and application of limiters is not necessary. However, HWENO, MP, GM, and TVD limiters have been applied in the smooth regions for the purpose of examining the performance and effect of the limiters in the smooth regions. The MP and HWENO limiters do not degrade the solution accuracy significantly even if these limiters are used without a trouble cell indicator. However, the small discrepancy between exact solution and HWENO limited solution may be observed

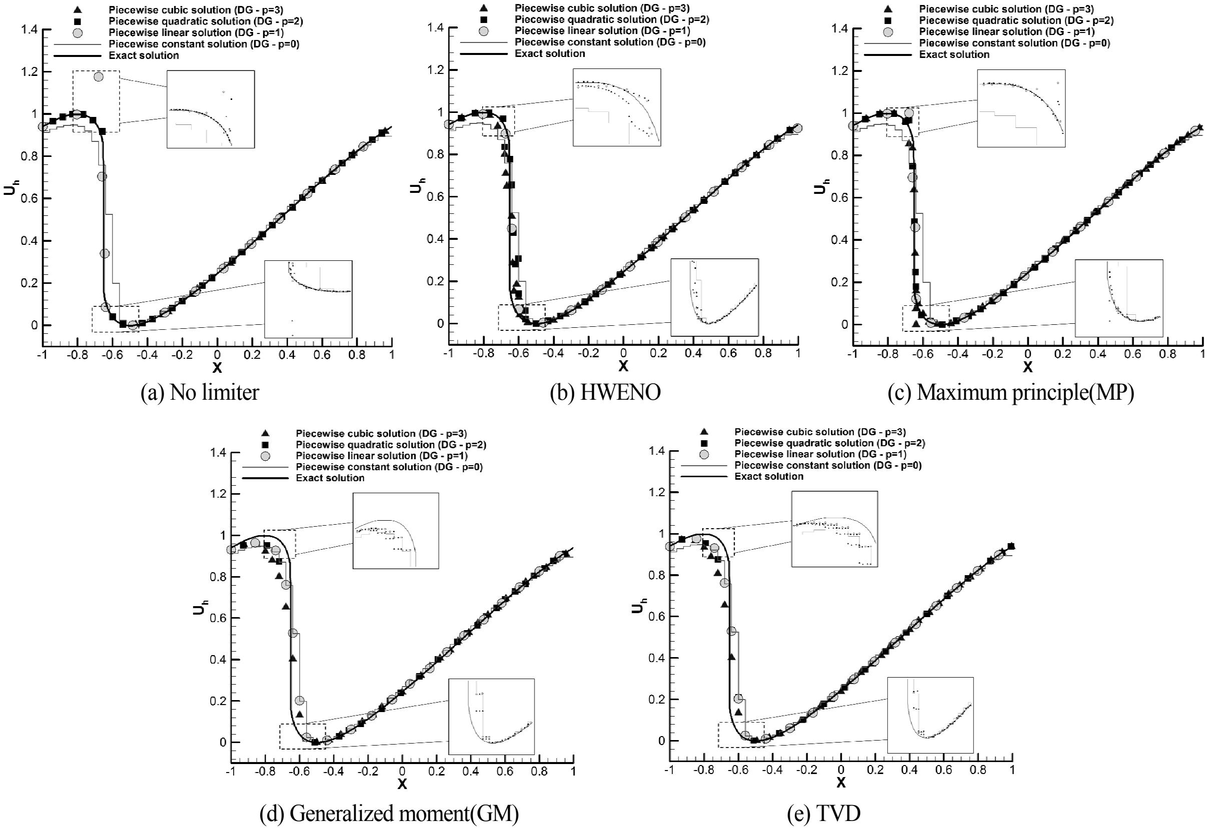


Fig. 3 Distribution of solution of the inviscid Burger's problem with sinusoidal initial distribution using 50 elements at $t = 0.7$ seconds

near both ends of the domain. On the other hand, application of TVD and GM limiters leads to a significant deviation from the exact solution. They cannot provide accurate results in smooth regions; therefore, application of a trouble cell indicator for detecting target elements is essential.

Inviscid Burger problem is considered as the next benchmark problem to investigate the total variation of the solution and its coefficients in smooth flow regions for different limiters functions. The periodic boundary condition and smooth sinusoidal initial distribution are used for this nonlinear scalar unsteady problem. The problem is given by

$$\partial_t u + u\partial_x u = 0, \ -1 \le x \le 1, t \ge 0,$$

$$u(x,0) = \frac{1+\sin(\pi x)}{2}, \tag{7}$$

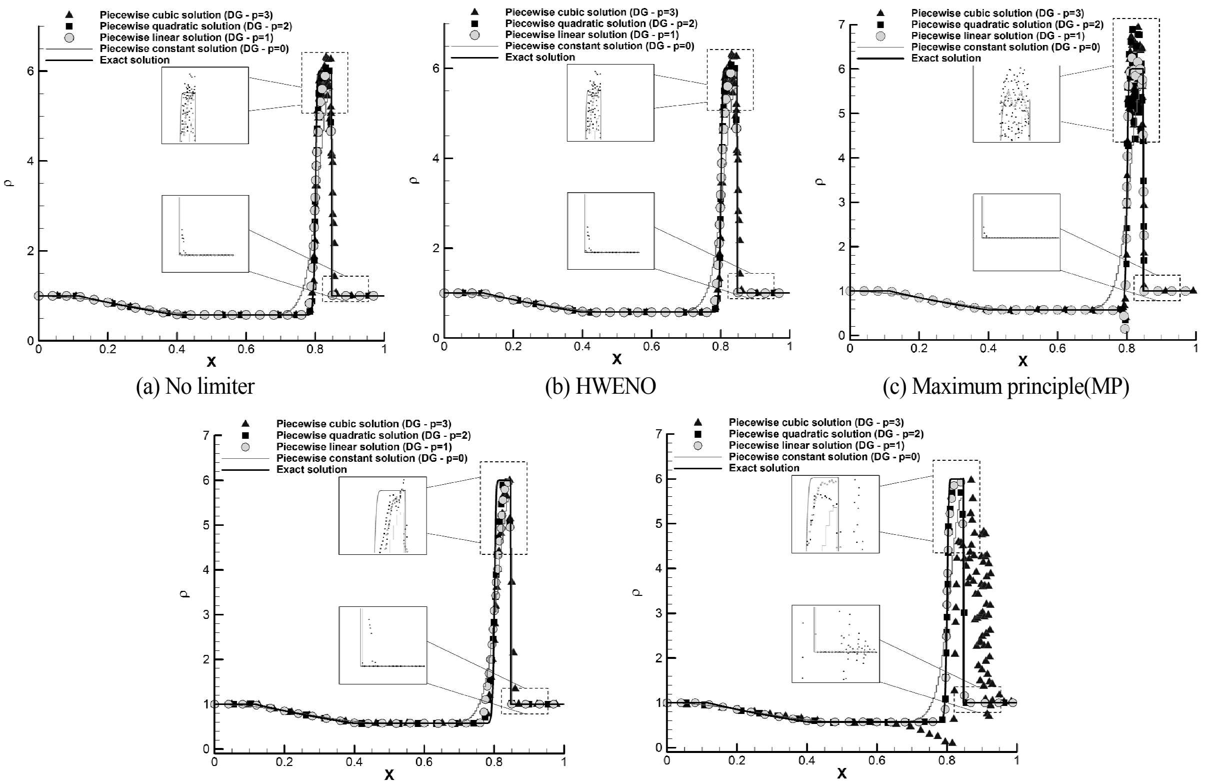
$$u(x_L,t) = u(x_R,t),$$

where x_L and x_R denote the left and right side of the computational domain.

Fig. 3 shows that the initial smooth profile transforms into the stiff profile with increasing time. Also, the high-order unlimited solutions remain oscillatory near the inflection point. In order to eliminate the spurious oscillations from the solutions, a spurious controller may be required.

Fig. 3 also shows that the Hermite WENO limiter can successfully preserve the accuracy of the solution. However, more number of the elements are usually required to capture the exact solution. The maximum principle limiter performs fairly, although it damps the oscillations insufficiently at the inflection point in the case of the piecewise linear approximation. The GM and TVD limiters show almost identical results.

In order to check the performance of limiters for a system of equations, the present modal RKDG method was applied for solving the Euler system. One-dimensional Riemann problem with an initial stationary contact



(a) No limiter (b) HWENO (c) Maximum principle(MP)

(d) Generalized moment(GM) (e) TVD

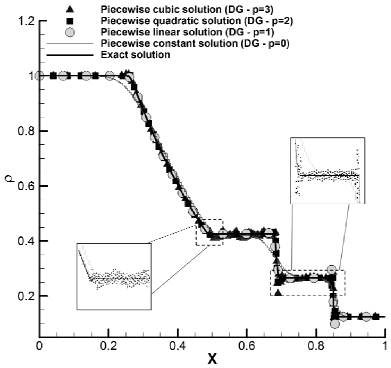
Fig. 4 The solutions of the stationary contact discontinuity problem at $t = 0.012$ seconds

discontinuity located at $x = 0.8$ is considered as a benchmark problem. Computational domain is discretized into 200 equal-sized elements and the simulation is run until the time reaches to $t = 0.012$ seconds.

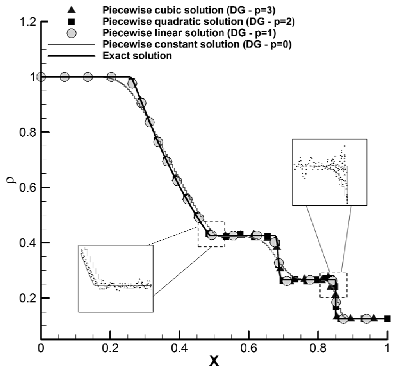
Fig. 4 shows the unlimited and limited numerical solutions. It can be seen that the unlimited solutions of the piecewise linear, quadratic, and cubic polynomial approximations are contaminated with a considerable amount of oscillations. The HWENO and MP limiters which performed well in the case of the linear and nonlinear advection problems, turns out not to be able to eliminate all spurious oscillations from the high-order solutions. Interestingly, the TVD and GM limiters, which performed poorly in the case of the linear and non-linear scalar hyperbolic problems, show better performance than the MP and WENO limiter for degree of $p \leq 2$ in the Euler system. This may be because the same linear weights were used for all problems in case of HWENO limiter. In contrast, in MP limiter, the local maximum and minimum were determined based on the initial values, rather than the adjacent values, which may have a significant effect on the performance for nonlinear systems.

As the next benchmark problem, we consider the Sod's shock tube flow that contains a left-running expansion wave, a contact discontinuity, and a right-running shock wave. In this problem, the computational domain is discretized with 200 elements and the simulation is run until the time reaches to $t = 0.2$ seconds.

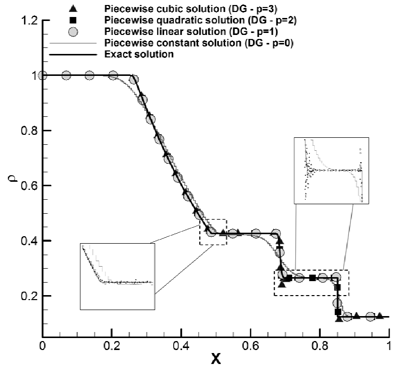
Fig. 5 shows the density profiles of the Sod's shock tube problem. In this case, the TVD limiter degrades the solution considerably. In particular, the TVD limiter for piecewise quadratic and cubic solutions($p = 2$, 3) yields unsatisfactory results. Similarly, the GM limiter suffers non-negligible wiggles in the case of the piecewise quadratic and cubic solutions. Interestingly, it gives non-oscillatory density profile in case of the piecewise linear solution. On the other hand, the MP limiter, which is free from user inputs, yields more accurate solution than the HWENO limiter. Nonetheless, it must be mentioned that obtaining an accurate solution using the MP limiter requires an accurate specification of the global maximum and minimum of the solution.



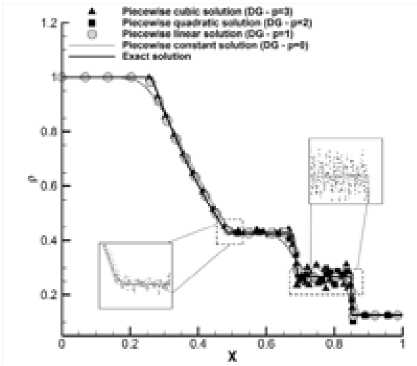
(a) No limiter



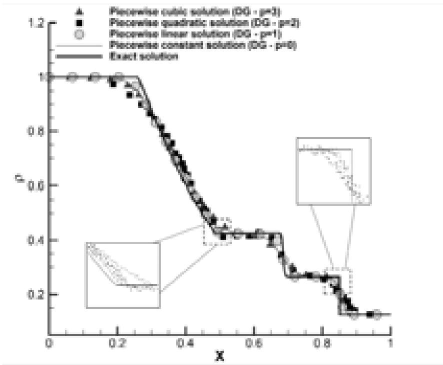
(b) HWENO



(c) Maximum principle(MP)

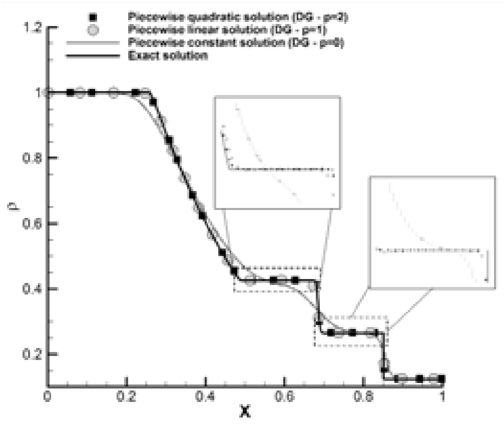


(d) Generalized moment(GM)

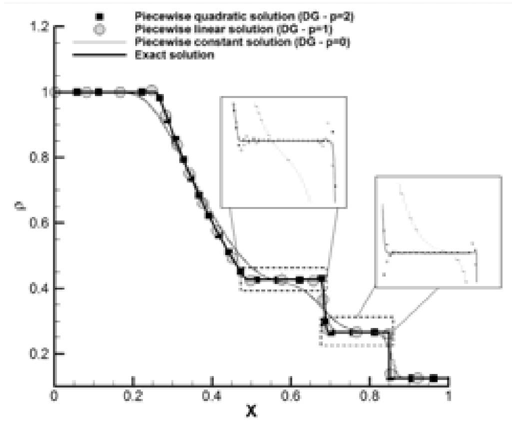


(e) TVD

Fig. 5 The solutions of the Sod's shock tube problem with 200 elements at $t = 0.2$ seconds



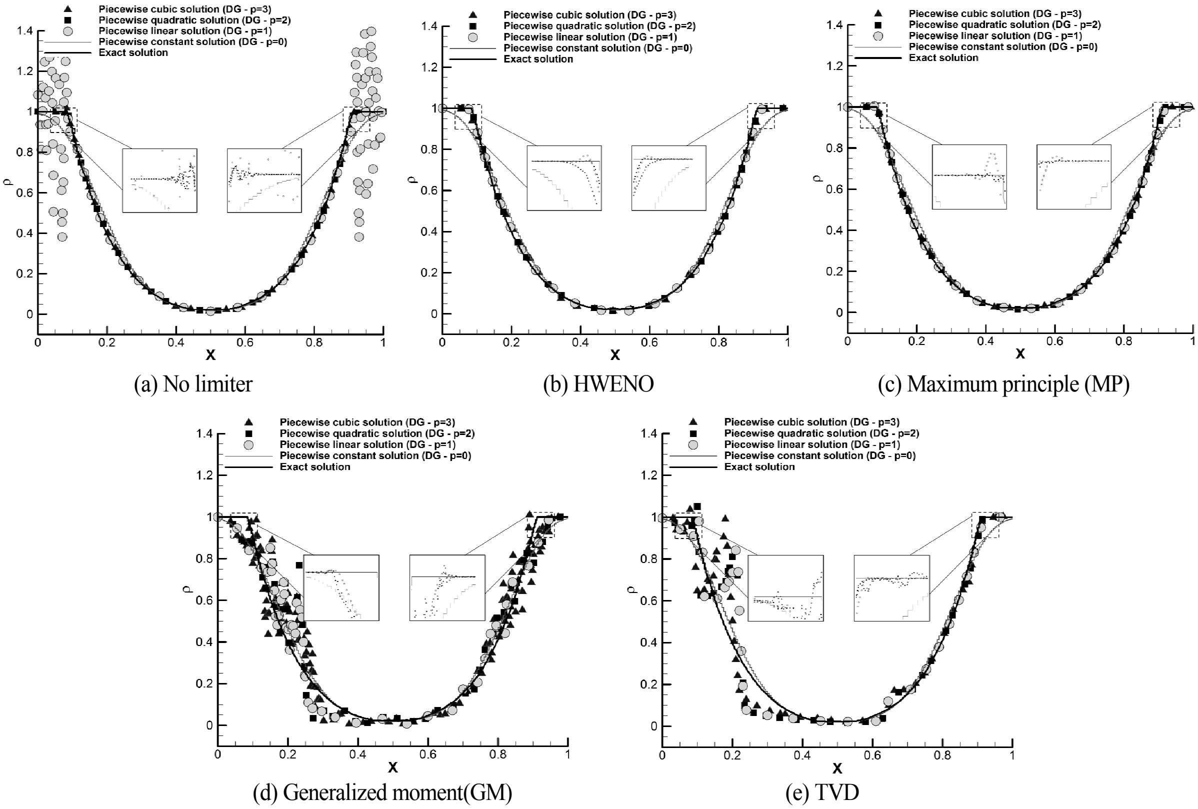
(a) Conservative field



(b) Characteristics field

Fig. 6 Comparison of the conservative and characteristics field variables in the limiting process of the HWENO limiter in conjunction with the KXRCF trouble cell indicator

In Fig. 6, the performance of conservative and characteristics field variables in the limiting process is compared. It can be seen that the combination of the HWENO limiter with the KXRCF trouble cell indicator and the characteristics field variables yields most accurate solutions. This result is related to the fact that application of the characteristic variables decouples one-dimensional nonlinear Euler system into a set of three simple wave



(a) No limiter (b) HWENO (c) Maximum principle (MP)

(d) Generalized moment(GM) (e) TVD

Fig. 7 The solutions of Einfeldt's strong rarefaction problem with 200 elements at $t = 0.15$ seconds

equations that describe the direction of the information propagation in the field. Characteristics wave-speeds and their directions isolate the local characteristics field in a way that preserves the wave structure of the Euler equations, and controls the upwind direction of the difference operators. Hence, in limiting process, the propagation characteristics of the flow are better taken into consideration. Nonetheless, it should be noted that the characteristics field variables cannot be employed in the case of Navier-Stokes-Fourier equations.

Two rarefaction wave propagation near vacuum[24] is the last studied problem. For this particular problem, the negative density and pressure may appear in the process of simulation. As a result, linearized Riemann solvers can fail by returning negative pressures or densities in one or more of the intermediate states for very strong rarefactions. Therefore, the positivity preserving limiter[21] is expected to play a critical role in the problem. Density profile solution of this problem is shown in Fig. 7. It is shown that the TVD and GM limiters degrade the accuracy of DG solution significantly. However, the MP and HWENO limiters in conjunction with positivity preserving feature can preserve the solution accuracy within an acceptable level.

6. Conclusion

In present work, a modal explicit DG method was developed for solving the 1-D hyperbolic systems including gas dynamic Euler equations. Several commonly used limiters were then applied to well-known high-speed benchmark flow problems. The unlimited and limited numerical solutions were compared to examine the performance of the limiters in detail. Results showed that the TVD and generalized moment(GM) limiters perform poorly in most of the cases. On the other hand, the HWENO and maximum principle limiters were shown to provide solutions with acceptable level of the accuracy. This is due to the fact that the HWENO scheme reconstructs a fifth-order non-oscillatory solution in the trouble element using the original DG high-order solutions and its adjacent neighbors. Also, in case of the maximum principle limiter, degradation of the accuracy of the solution is minimized through relaxation of the monotonicity condition and activation of limiter only when the high-order solution in the trouble element exceeds the maximum and minimum value of the initial condition. It was also shown that application of characteristics field variables in the limiting process yields better solution, owing to the feature that the propagation characteristics of the flow are better taken into consideration.

The present study also showed that the maximum principle limiter may be considered reliable if the global maximum and minimum variables are defined precisely. Furthermore, it was shown that the limiters with troubled cell indicators in general provide better solutions. On the other hand, the TVD, generalized moment limiters without employing a trouble cell indicator led to significant degradation of the solution accuracy.

The present study has been limited to the linear and non-linear hyperbolic systems without the diffusive viscous terms. The extension of the present line of investigation on limiters to the case of convection-diffusion problem like the Navier-Stokes- Fourier equations will require additional efforts.

Further, only one-dimensional problems have been analyzed for their simplicity, although most of the practical applications are multi-dimensional and thus accurate multi-dimensional limiters remain critical. Nevertheless, we expect that the results obtained from the present work will remain important in analyzing and designing high performance limiters, since the essential theoretical and numerical properties of shock discontinuities are well-defined in one-dimensional situation.

Acknowledgments

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