Pushing the Limits of Continuum Fluid Mechanics and Going beyond the Navier-Stokes-Fourier

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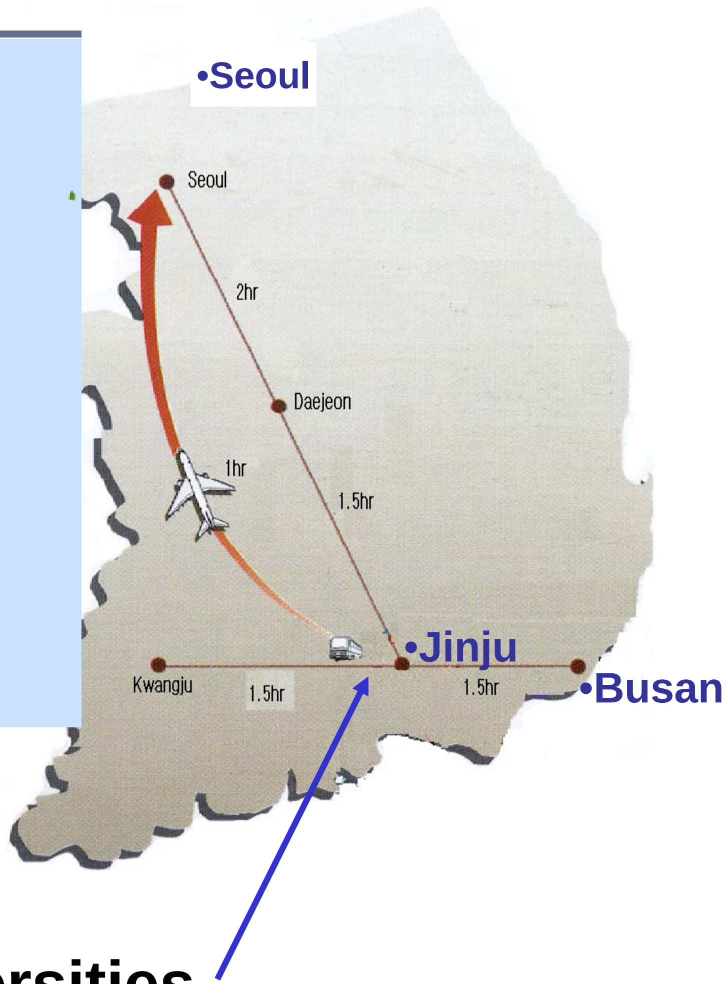
Introduction to Gyeongsang National University

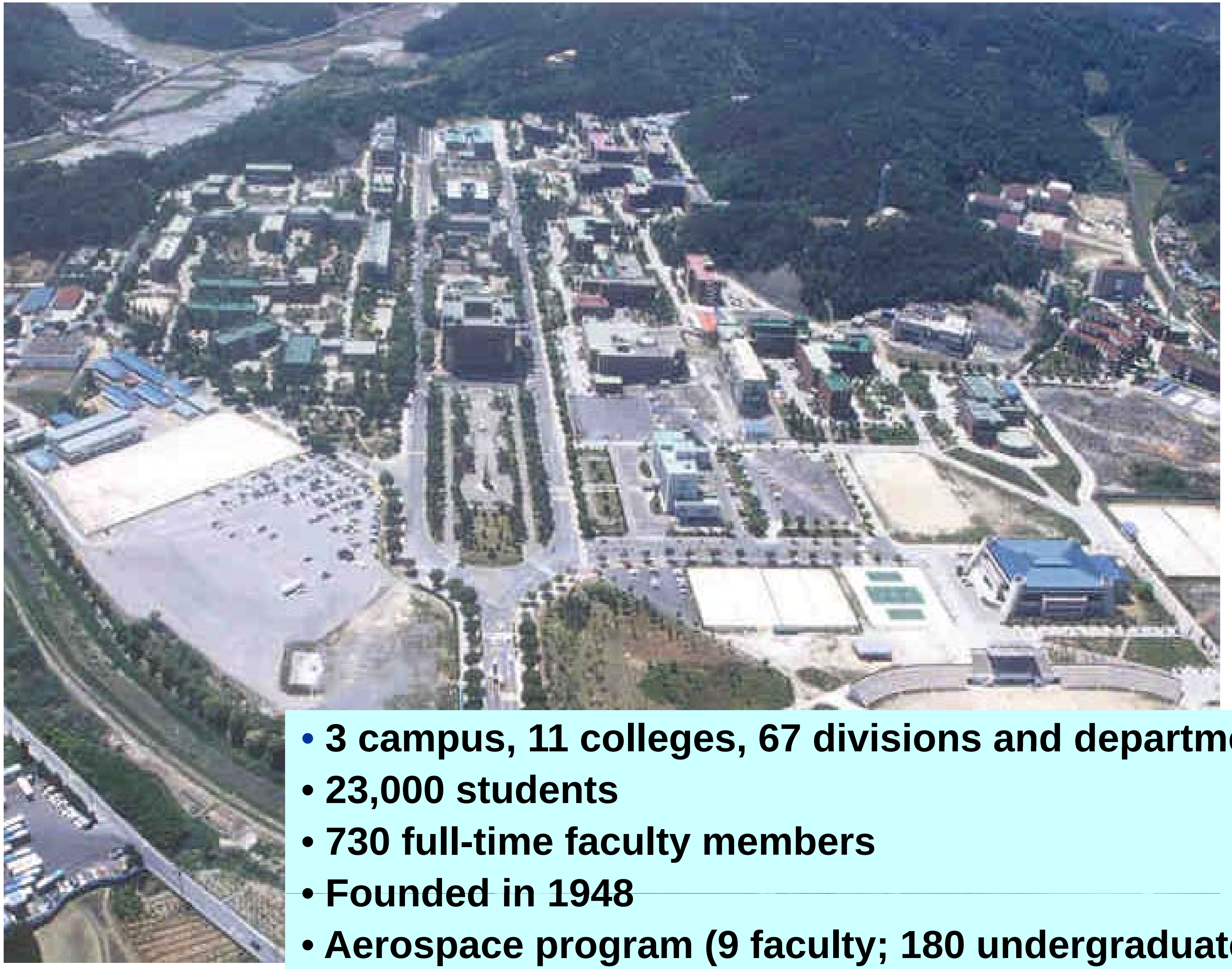


9 Provinces in South Korea

One of the major national universities

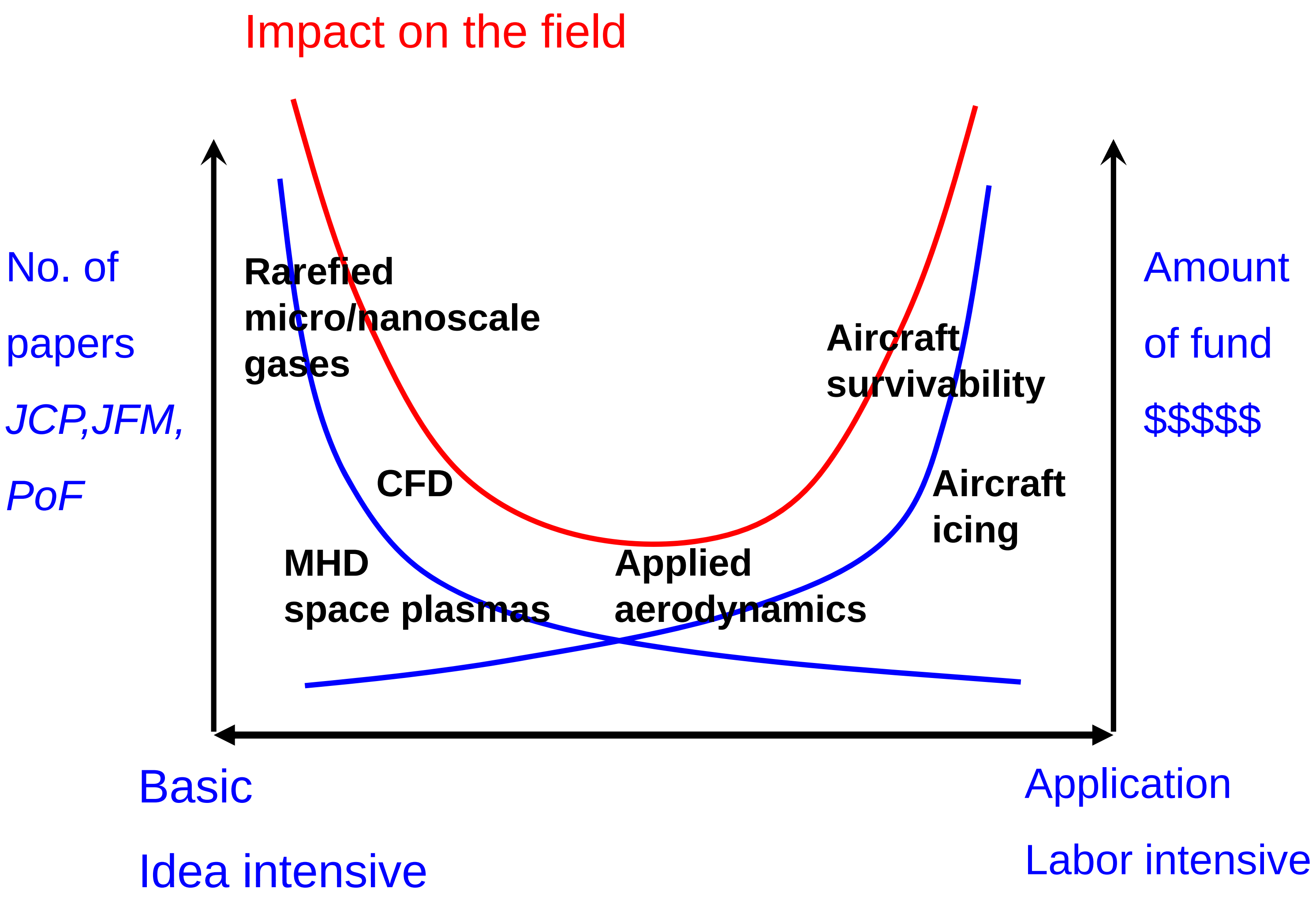
Located in Gyeongnam province (manufacturing industrial belt)

Overview of GNU



- 3 campus, 11 colleges, 67 divisions and departments
- 23,000 students
- 730 full-time faculty members
- Founded in 1948
- Aerospace program (9 faculty; 180 undergraduates; 120 graduates)

Research areas of Aerospace Comp. Modeling Lab



Research goal:

Develop a unified computational model for rarefied and micro- & nano-scale gases upon which others can build efficient CFD codes

http://acml.gnu.ac.kr ⇨ <Open knowledge>

Part I.

Fundamentals

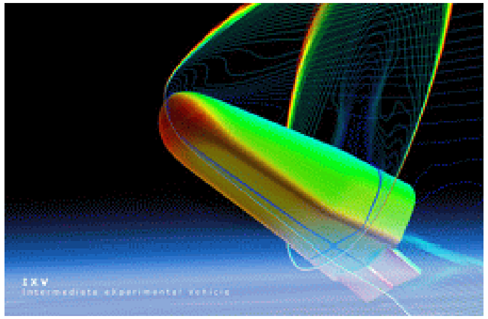
Introduction to rarefied and micro/nanoscale gases

Compression-dominated

High *M*, low Kn

Need of unified framework

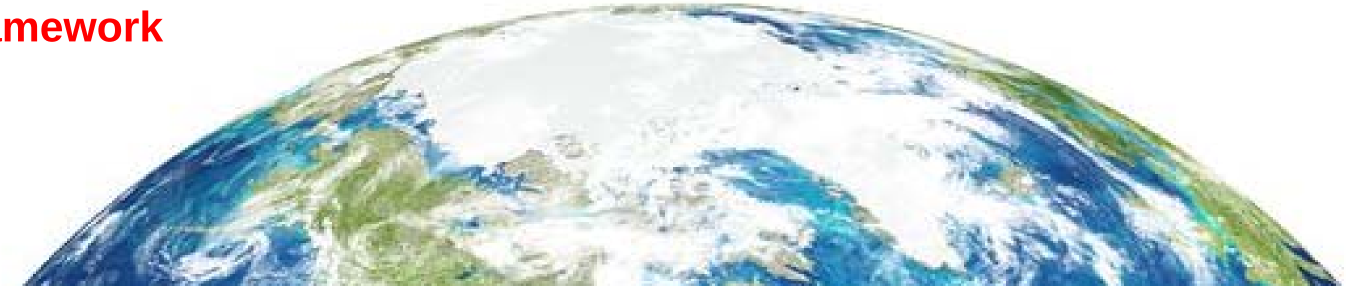
Intermediate Experimental Vehicle



Gas flows around hypersonic vehicles and plume flows

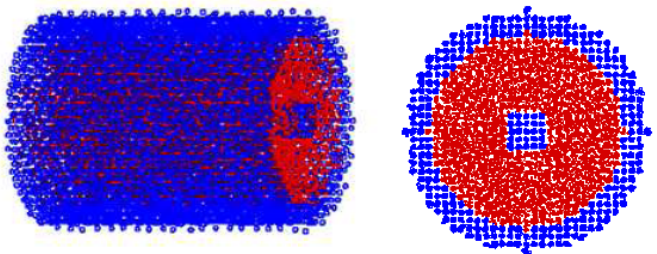
Continuous shift of continuum, transition, and free-molecular regimes

Coexistence of various regimes



Shear-dominated

Low *M*, high Kn



Micro and nanoscale cylinder

Gas (liquid) flow + MN solid devices

Molecular interaction between gas (liquid) particles and solid atoms

Gas (liquid) flows in thermal (trans., rot.) nonequilbrium regimes

Electrokinetics, surface tension etc.

Approaches for modeling rarefied and micro/nanoscale gases

Molecular approach

DSMC (Direct Simulation Monte Carlo) (Bird)

Linearized Boltzmann equation (Cercignani, Sone)

Lattice-Boltzmann method

Continuum approach

Chapman-Enskog: Burnett (1935) etc.

Moment method: Grad (1949), Eu (1992), regularized-13 (2005)

Constitutive equations: the only ingredient in the conservation laws in which the microscopic nature of gas molecules is taken into account.

Hybrid approach

DSMC-continuum coupling (Nie 2004, Schwartzentruber 2007)

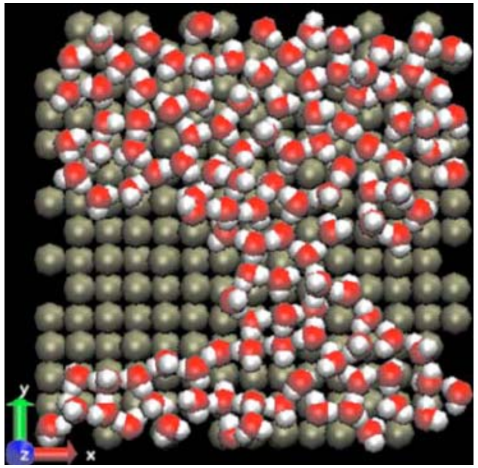
Seamless multi-scale method: viscous stress calculated by MD (Weinan 2009)

Modeling micro and nanomechanics of fluids and rarefied gases

Top-down: the classical linear (fluid mechanics) theories can account for virtually everything about materials (fluids).

Bottom-up: only a molecular-statistical theory of the structure of fluids can provide understanding of their true behavior.

$$\rho \frac{D}{Dt}\begin{bmatrix} 1/\rho \\ \mathbf{u} \\ E_t \end{bmatrix} + \nabla \cdot \begin{bmatrix} \mathbf{u} \\ p\mathbf{I} + \mathbf{\Pi} \\ \left(p\mathbf{I} + \mathbf{\Pi}\right) \cdot \mathbf{u} + \mathbf{Q} \end{bmatrix} = \mathbf{0}$$



$$\mathbf{\Pi} = -\eta \left[\nabla \mathbf{u}\right]^{(2)}, \ \mathbf{Q} = -k \nabla T$$

Navier **Fourier**

Linear uncoupled

constitutive relations

A critical observation: an efficient way of including the molecular nature of gases is to develop nonlinear coupled constitutive relations but to retain the conservation laws.

Linear uncoupled Navier-Fourier equations

Navier (1822)

Fourier (1822)

$$\rho \frac{D\mathbf{u}}{Dt} + \nabla p + \nabla \cdot \mathbf{\Pi} = \mathbf{0}, \quad \rho \frac{DE_t}{Dt} + \nabla \cdot p\mathbf{u} + \nabla \cdot \left[\mathbf{\Pi} \cdot \mathbf{u} + \mathbf{Q} \right] = \mathbf{0}$$

$$\mathbf{\Pi} = -\eta \left[\nabla \mathbf{u} \right]^{(2)}, \qquad \mathbf{Q} = -k \nabla T$$

$$\begin{bmatrix} \Pi_{xx} & \Pi_{xy} \\ \Pi_{yx} & \Pi_{yy} \end{bmatrix} \leftarrow -2\eta \begin{bmatrix} u_x - (u_x + v_y)/3 & (u_y + v_x)/2 \\ (v_x + u_y)/2 & v_y - (u_x + v_y)/3 \end{bmatrix}$$

u_x case (compression and expansion)

$$-2\eta \begin{bmatrix} 2u_x/3 & 0 \\ 0 & -u_x/3 \end{bmatrix}$$

Not like $(u_x)^2$ **Newtonian or linear**

u_y case (velocity shear only)

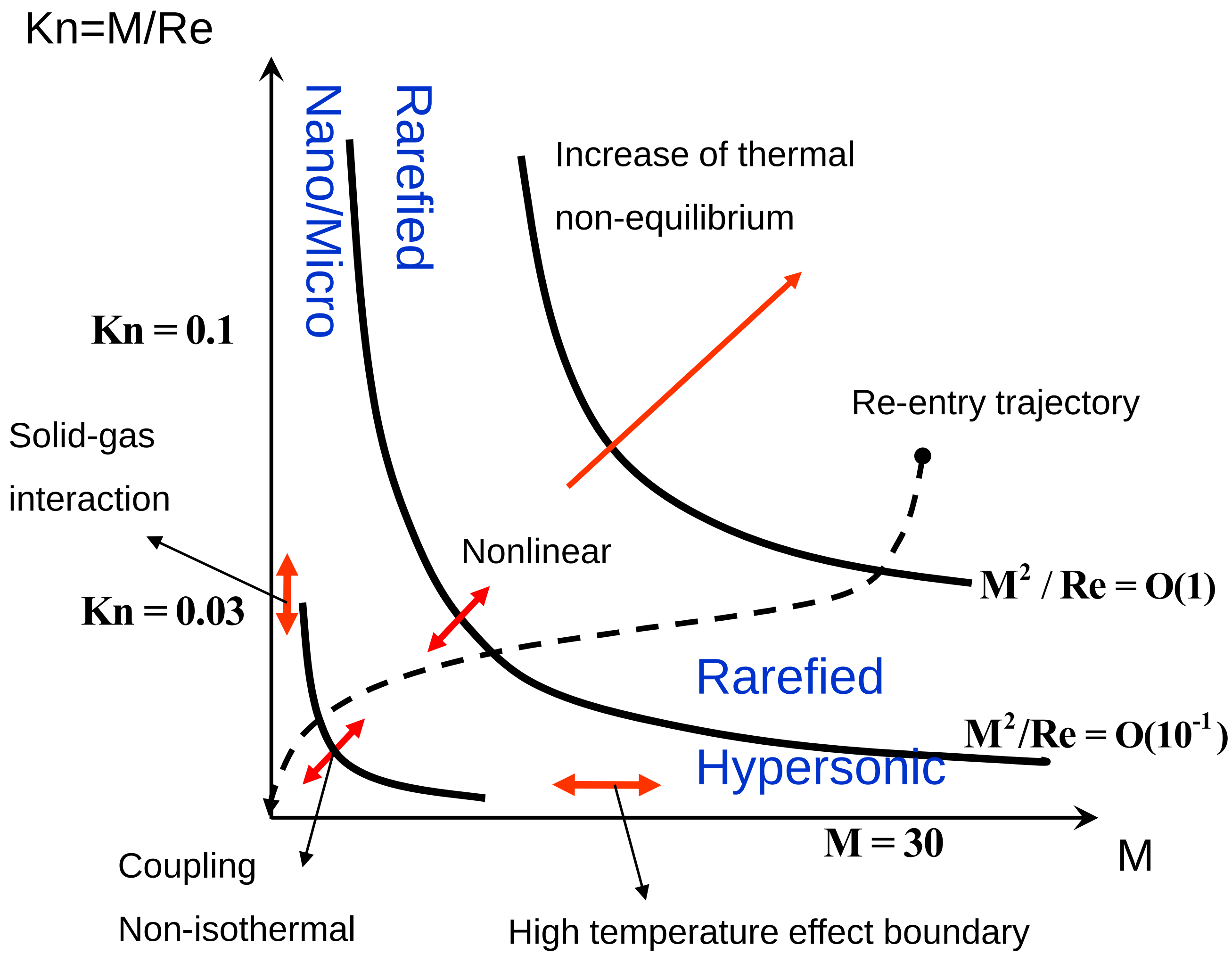
$$-2\eta \begin{bmatrix} 0 & u_y/2 \\ u_y/2 & 0 \end{bmatrix}$$

Uncoupled

Always vanishing normal stress Π_{xx} or Π_{yy}

$$\begin{bmatrix} Q_x \\ Q_y \end{bmatrix} \leftarrow -k \begin{bmatrix} T_x \\ T_y \end{bmatrix}. \quad \text{Not like } (T_x)^2$$

Physics of rarefied and micro/nanoscale gases



$\mathbf{v} \cdot \nabla f(\mathbf{r}, \mathbf{v}) = C[f, f_2]$

Two terms: Kn

$\rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla \cdot p\mathbf{I} + \nabla \cdot \mathbf{\Pi} = \mathbf{0}$

Three terms: *M*, Kn

$\Longrightarrow$ **Main parameter** $\Pi / p \sim \mathrm{Kn} \cdot M$

(not Kn alone!)

Modelling of nonequilibrium gas system

Molecular (Probabilistic) **Phase Space** **Boltzmann (1844-1906)**

$$f\,(t,\mathbf{r};\mathbf{v})$$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla+\mathbf{a}\cdot\nabla_v\right)f\,(t,\mathbf{r};\mathbf{v})=C\left[f\,,\,f_2\right]$$

Thermodynamics

(Reduction of

Information)

$$\beta=\frac{1}{k_B T}$$

Statistical average

$$\rho=\left\langle mf\,(t,\mathbf{r};\mathbf{v})\right\rangle$$

$$\rho\mathbf{u}=\left\langle m\mathbf{v}f\,(t,\mathbf{r};\mathbf{v})\right\rangle$$

$$\langle\cdots\rangle=\iiint\cdots dv_x dv_y dv_z$$

$$\left(\rho,\mathbf{u},T,\Pi,\mathbf{Q},\cdots\right)(t,\mathbf{r})$$

$$\rho\frac{D\mathbf{u}}{Dt}+\nabla\cdot\left(p\mathbf{I}+\Pi\right)=\rho\mathbf{a}$$

Continuum

(Hydrodynamic)

Thermodynamic

Space

Conservation Laws

Moment Equation

(Constitutive Relation)

Not far from LTE

Navier-Stokes-Fourier

A modified moment method [Eu, 1992]

$$\rho \equiv \left\langle mf(t,\mathbf{r};\mathbf{v})\right\rangle, \quad \rho\mathbf{u} \equiv \left\langle m\mathbf{v}f(t,\mathbf{r};\mathbf{v})\right\rangle, \quad \left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \mathbf{a}\cdot\nabla_v\right) f(t,\mathbf{r};\mathbf{v}) = C\left[f, f_2\right]$$

Differentiating the statistical definition $\rho \equiv \left\langle mf(t,\mathbf{r};\mathbf{v})\right\rangle$ *with time* and *then combining* with the Boltzmann equation

$$\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial t}\left\langle mf(t,\mathbf{r};\mathbf{v})\right\rangle = \left\langle m\frac{\partial f}{\partial t}\right\rangle = \left\langle mC\left[f, f_2\right]\right\rangle - \left\langle m\mathbf{v}\cdot\nabla f\right\rangle$$

$$\frac{\partial \rho}{\partial t} + \left\langle m\mathbf{v}\cdot\nabla f\right\rangle = \left\langle mC\left[f, f_2\right]\right\rangle = 0$$

$$\frac{\partial \rho}{\partial t} + \left\langle m\nabla\cdot\left(f\mathbf{v}\right)\right\rangle - \left\langle mf\nabla\cdot\mathbf{v}\right\rangle = \frac{\partial \rho}{\partial t} + \left\langle m\nabla\cdot\left(f\mathbf{v}\right)\right\rangle = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left\langle mf\mathbf{v}\right\rangle = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\mathbf{u}\right) = 0$$

The modified moment method (continued)

Moment equations in vector form

$$\rho \frac{D}{Dt}\begin{bmatrix} 1/\rho \\ \mathbf{u} \\ E_t \end{bmatrix} + \nabla \cdot \begin{bmatrix} \mathbf{u} \\ p\mathbf{I} \\ p\mathbf{u} \end{bmatrix} + \nabla \cdot \begin{bmatrix} 0 \\ \mathbf{\Pi} \\ \mathbf{\Pi}\cdot\mathbf{u} + \mathbf{Q} \end{bmatrix} = \begin{bmatrix} 0 \\ \rho\mathbf{a} \\ \rho\mathbf{a}\cdot\mathbf{u} \end{bmatrix}$$

$$\left(\partial_t + \mathbf{v}\cdot\nabla + \mathbf{a}\cdot\nabla_v\right) f(t,\mathbf{r};\mathbf{v}) = C\left[f, f_2\right]$$

Unsteady **Convection** **Higher-order**

$$\frac{\partial}{\partial t}\begin{bmatrix} \mathbf{\Pi} \\ \mathbf{Q} \end{bmatrix} + \mathbf{u}\cdot\nabla\begin{bmatrix} \mathbf{\Pi} \\ \mathbf{Q} \end{bmatrix} = -\begin{bmatrix} \nabla\cdot\boldsymbol{\psi}^{(\Pi)} \\ \nabla\cdot\boldsymbol{\psi}^{(Q)} + \boldsymbol{\psi}^{(P)} : \nabla\mathbf{u} \end{bmatrix}$$

$$-\begin{bmatrix} 2\left[\mathbf{\Pi}\cdot\nabla\mathbf{u}\right]^{(2)} \\ C_p\mathbf{\Pi}\cdot\nabla T + \mathbf{Q}\cdot\nabla\mathbf{u} + \frac{D\mathbf{u}}{Dt}\cdot\mathbf{\Pi} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{a}\cdot\mathbf{\Pi} \end{bmatrix} - \begin{bmatrix} 2p\left[\nabla\mathbf{u}\right]^{(2)} \\ C_p p\nabla T \end{bmatrix} + \begin{bmatrix} \mathbf{\Lambda}^{(\Pi)} \\ \mathbf{\Lambda}^{(Q)} \end{bmatrix}$$

Kinematic **Force** **Thermo. driving** **Dissipation**

$$\mathbf{\Pi} \equiv \left\langle m\left[\mathbf{cc}\right]^{(2)} f\right\rangle, \quad \mathbf{Q} \equiv \left\langle mc^2\mathbf{c}f/2\right\rangle,$$

$$\boldsymbol{\psi}^{(\Pi)} \equiv \left\langle m\left[\mathbf{cc}\right]^{(2)}\mathbf{c}f\right\rangle, \quad \boldsymbol{\psi}^{(P)} \equiv \left\langle m\mathbf{ccc}f\right\rangle, \quad \boldsymbol{\psi}^{(Q)} \equiv \left\langle mc^2\mathbf{cc}f/2\right\rangle$$

$$\mathbf{\Lambda}^{(\Pi)} \equiv \left\langle m\left[\mathbf{cc}\right]^{(2)} C[f, f_2]\right\rangle, \mathbf{\Lambda}^{(Q)} \equiv \left\langle mc^2\mathbf{c}/2C[f, f_2]\right\rangle$$

A physically motivated closure

By noting the relative importance of various terms, for example,

Higher-order

$$\left|\nabla\cdot\boldsymbol{\psi}^{(\Pi)}\right| < \left|\left[\boldsymbol{\Pi}\cdot\nabla\mathbf{u}\right]^{(2)}\right|, \left|p\left[\nabla\mathbf{u}\right]^{(2)}\right|$$

Kinematic **Thermo. driving**

one may neglect the higher-order term as an approximation.

$$\nabla\cdot\boldsymbol{\psi}^{(\Pi)} \approx 0,$$

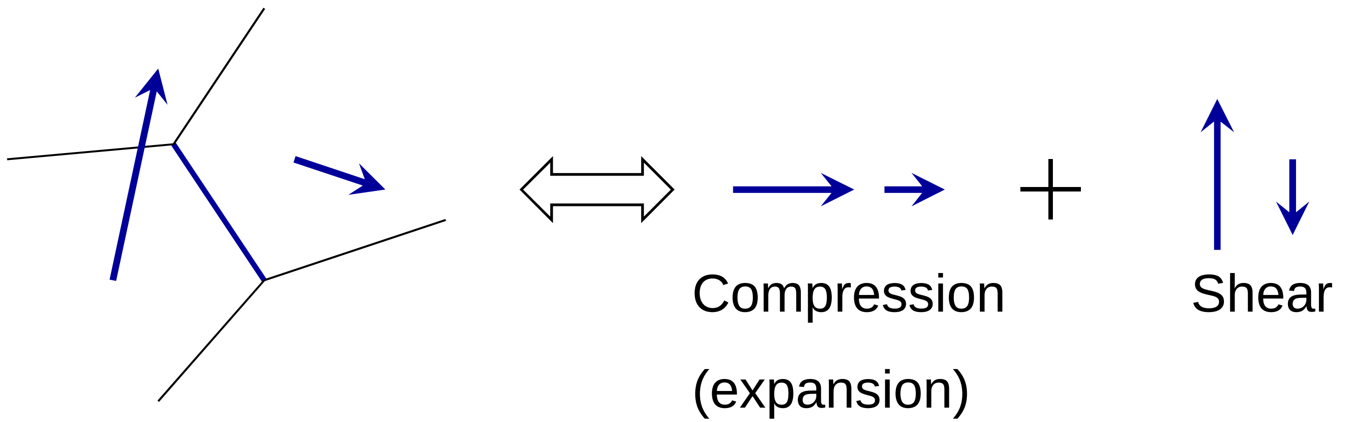
$$\nabla\cdot\boldsymbol{\psi}^{(Q)} + \boldsymbol{\psi}^{(P)} : \nabla\mathbf{u} \approx 0$$

Then we have a nonlinear coupled algebraic constitutive relation (NCCR)

$$0 = -\begin{bmatrix} 2\left[\boldsymbol{\Pi}\cdot\nabla\mathbf{u}\right]^{(2)} \\ C_p\boldsymbol{\Pi}\cdot\nabla T + \mathbf{Q}\cdot\nabla\mathbf{u} + \dfrac{D\mathbf{u}}{Dt}\cdot\boldsymbol{\Pi} \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{a}\cdot\boldsymbol{\Pi} \end{bmatrix} - \begin{bmatrix} 2p\left[\nabla\mathbf{u}\right]^{(2)} \\ C_p p\nabla T \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Lambda}^{(\Pi)} \\ \boldsymbol{\Lambda}^{(Q)} \end{bmatrix}$$

Kinematic **Force** **Thermo. driving** **Dissipation**

A computational framework based on NCCR



Edge based finite volume formulation in general coordinates (*JCP* 2004)

$$\frac{\partial}{\partial t}\int_V \mathbf{U}\,dV + \int_S \mathbf{F}\cdot\mathbf{n}\,dS = 0$$

$$\mathbf{U}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n} - \frac{\Delta t}{A_{i,j}}\sum_{k=1}^{N}\mathbf{R}_k^{-1}\mathbf{F}_k^{n}\Delta L_k$$

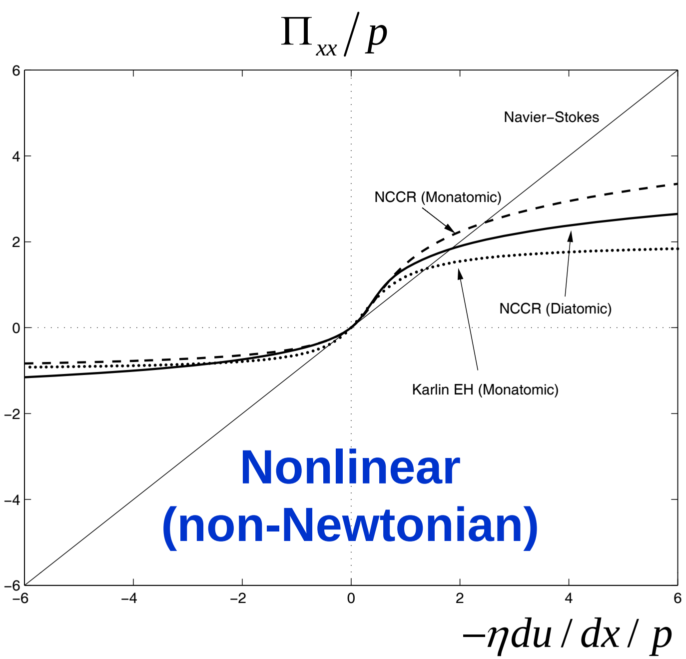
$$\mathbf{\Pi} = f_{\Pi}\left(\mathbf{\Pi}_{\mathrm{NSF}},\mathbf{Q}_{\mathrm{NSF}},p,T\right),\quad \mathbf{Q} = f_{Q}\left(\mathbf{\Pi}_{\mathrm{NSF}},\mathbf{Q}_{\mathrm{NSF}},p,T\right)$$

NCCR property in compression-expansion and velocity shear case

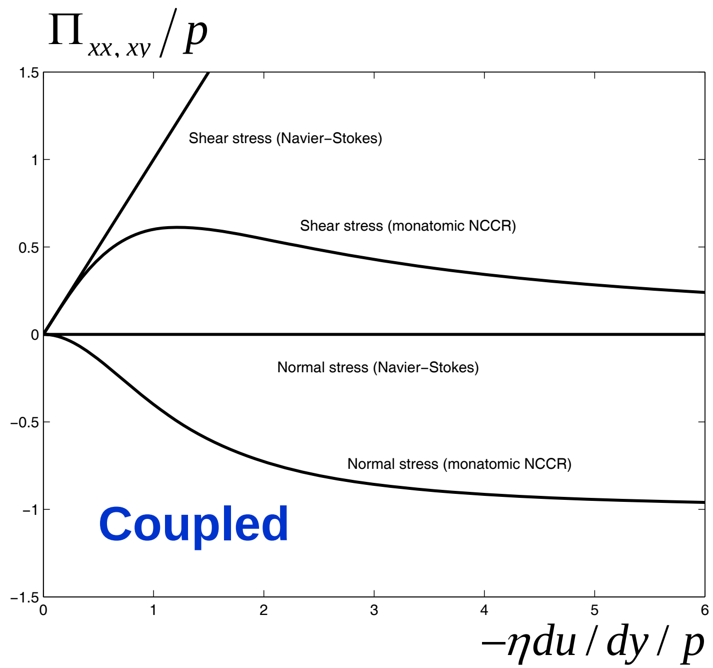
$$-2\eta\left[\nabla \mathbf{u}\right]^{(2)} \quad \xrightarrow{\div p} \quad \boxed{\text{Algebraic NCCR}} \quad \xrightarrow{\times p} \quad \mathbf{\Pi}$$

$$-k\nabla T \quad \xrightarrow{\div p\sqrt{\frac{C_p T}{2\,\mathrm{Pr}}}} \quad \boxed{\text{Algebraic NCCR}} \quad \xrightarrow{\times p\sqrt{\frac{C_p T}{2\,\mathrm{Pr}}}} \quad \mathbf{Q}$$

Thermodynamic driving force → **Algebraic NCCR** → **Shear stress heat flux**



expansion **compression**



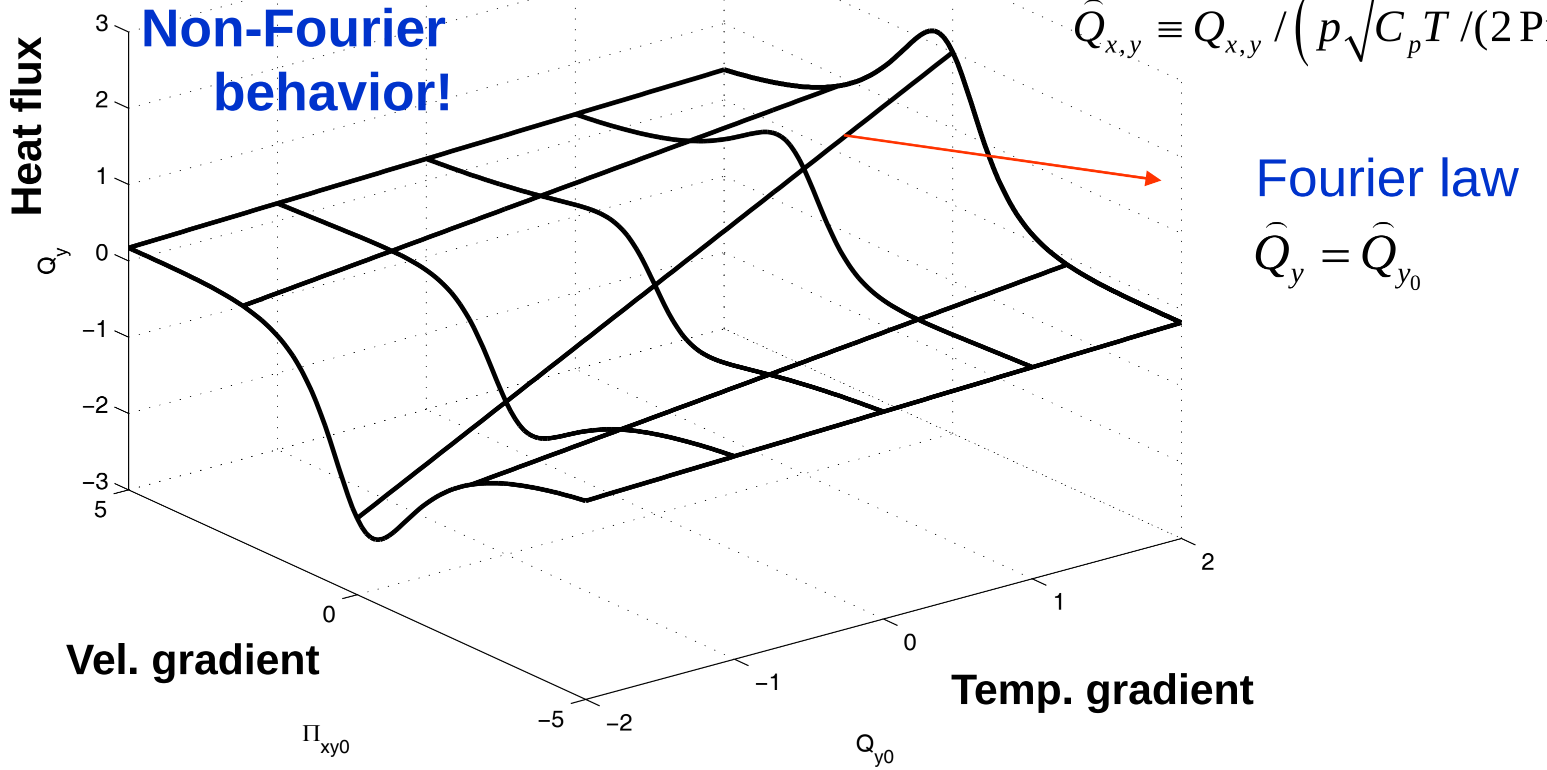
velocity shear

Non-Fourier law by the coupling of force and shear stress

$$\widehat{Q}_y = \frac{3}{\left(3+2\widehat{\Pi}^2_{xy_0}\right)}\left(\widehat{Q}_{y_0} + \widehat{a}\widehat{\Pi}_{xy_0}\right);\ a \text{ is force}$$

$$\hat{\Pi}_{xy} \equiv \hat{\Pi}_{xy} / p$$

$$\widehat{Q}_{x,y} \equiv Q_{x,y} / \left(p\sqrt{C_p T/(2\,\mathrm{Pr})}\right)$$



Part II.

Validation

Validation I: Celebrated shock structure

Major stumbling blocks for theoreticians for a long time

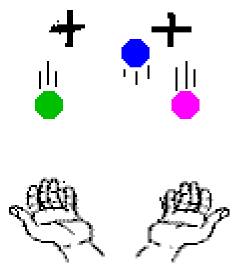
"… *is found that the solution breaks down completely, and no solution exists for stronger shocks (specifically, at Mach number M=1.65)*…" (H. Grad 1952)

It was solved by B. C. Eu in 1997 (*Phys. Rev. E*)

How his theory works is explained here.

Graveyard of theoreticians

Validation I (continued)



Unsteady   Convection   Higher-order

$$\frac{\partial \mathbf{\Pi}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{\Pi} = -\nabla\cdot\boldsymbol{\psi}^{(\Pi)} - 2\left[\mathbf{\Pi}\cdot\nabla\mathbf{u}\right]^{(2)} - 2p\left[\nabla\mathbf{u}\right]^{(2)} + \mathbf{\Lambda}^{(\Pi)}$$

Kinematic   Thermo. driving   Dissipation

Comment: In steady-state case, one has to deal with 5 terms, which is really difficult juggling. Even in simple 3 ball juggling, there can be as many as 17 methods.

Key finding: Gases are compressed rapidly across the shock wave and so accurate treatment of dissipation term will be critical.

What B. C. Eu did: Assume the distribution function *f* in exponential form (not in usual polynomial form) and then apply the cumulant expansion method: the celebrated Eu's hyperbolic sine factor.

Validation I (continued)

Essential terms are three in the right-hand side

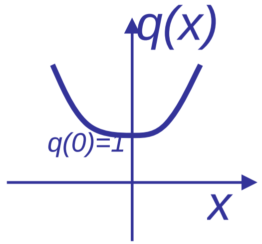
$$0 + 0 = 0 - 2\left[\mathbf{\Pi}\cdot\nabla\mathbf{u}\right]^{(2)} - 2p\left[\nabla\mathbf{u}\right]^{(2)} + \mathbf{\Lambda}^{(\Pi)}$$

Kinematic **Thermo. driving** **Dissipation**

$$\mathbf{\Lambda}^{(\Pi)} = -\frac{\mathbf{\Pi}}{\eta / p} q(\kappa), \quad \boxed{q(\kappa) \equiv \frac{\sinh\kappa}{\kappa}} \quad \text{where}\ \kappa = \frac{(mk_B T)^{1/4}}{\sqrt{2}pd}\left(\frac{\mathbf{\Pi}:\mathbf{\Pi}}{2\eta} + \frac{\mathbf{Q}\cdot\mathbf{Q}}{kT}\right)^{1/2}$$

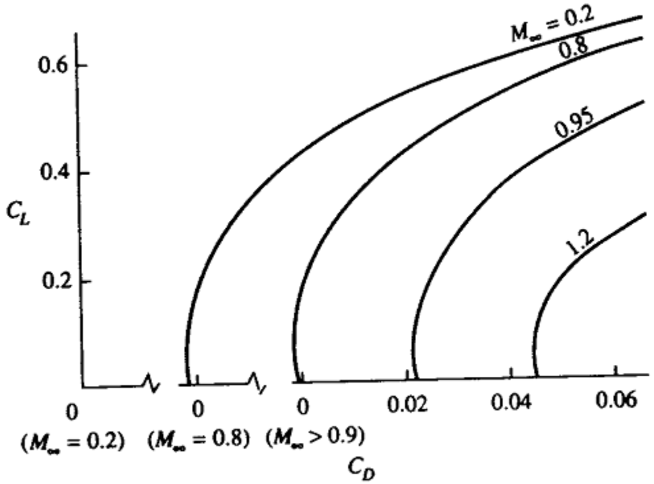
Cf. q(kappa)=1; BGK or relaxation approx.

$$C[f, f_2] \approx \frac{f^{equil} - f}{\tau}$$



Comment: No 1st-order term in the series expansion of q(kappa) explains why the NSF approximation is so successful.

Cf. Drag polar in aerodynamics



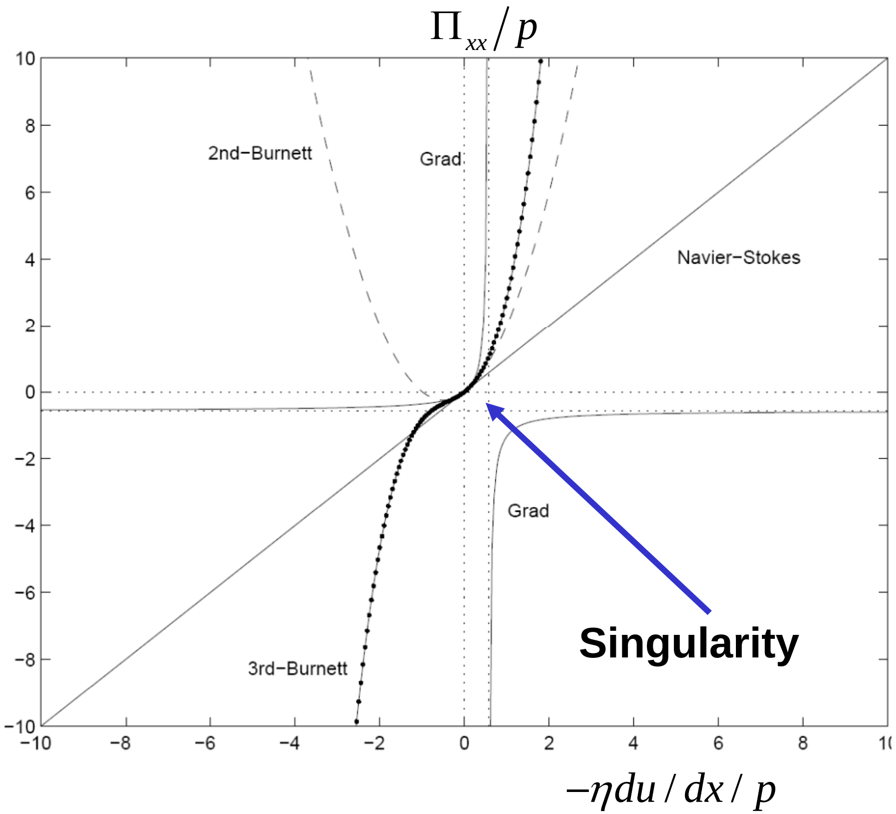
Validation I (continued)

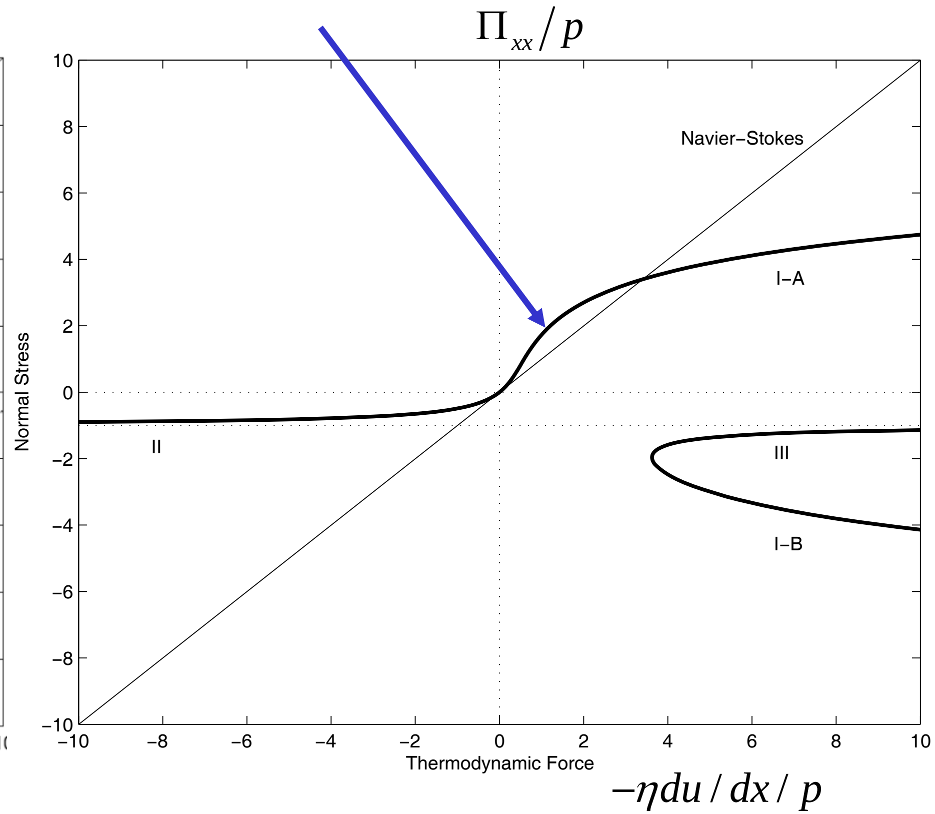
Before

With $q(\kappa) \equiv \frac{\sinh \kappa}{\kappa}$

After

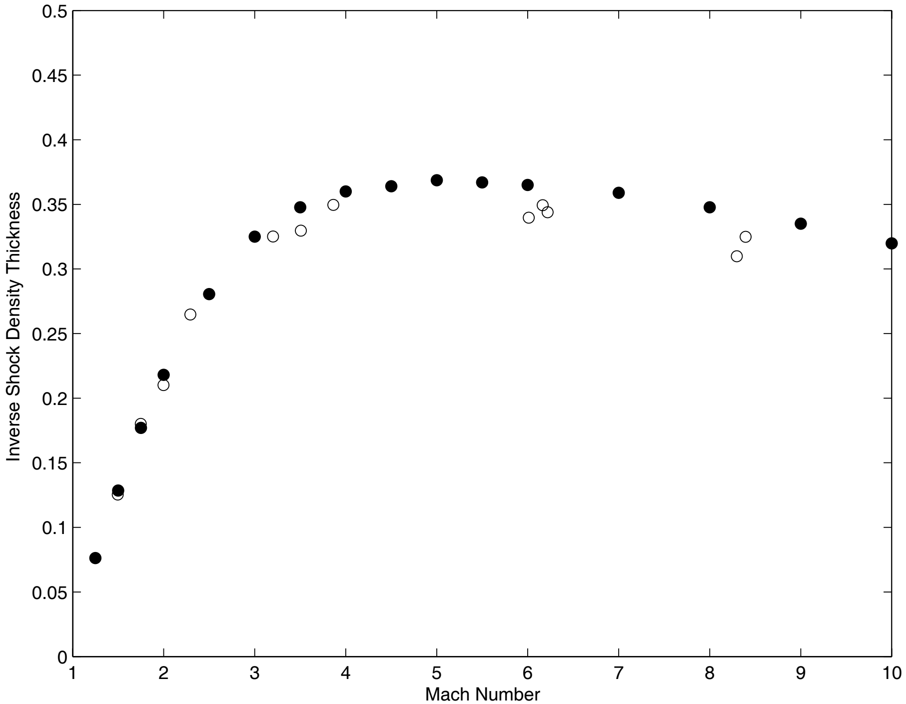
singularity removed!



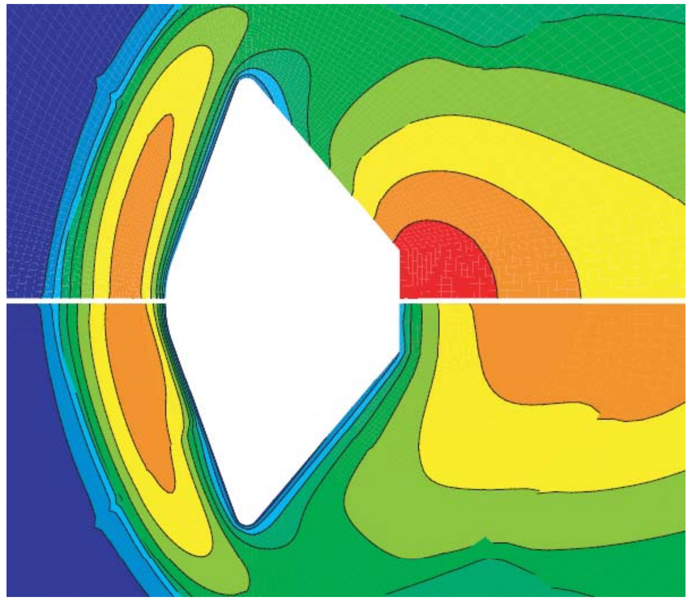


expansion **compression** **expansion** **compression**

Validation I (continued)

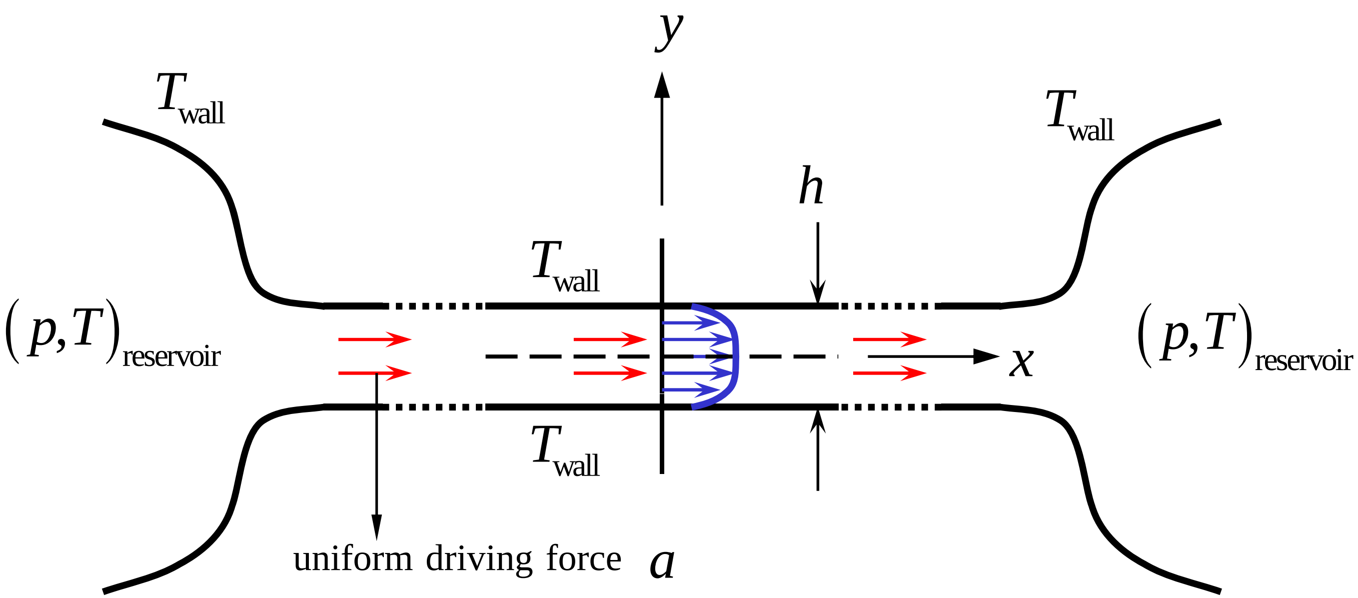


Shock density thickness for a diatomic gas versus the Mach number (o: experimental data)



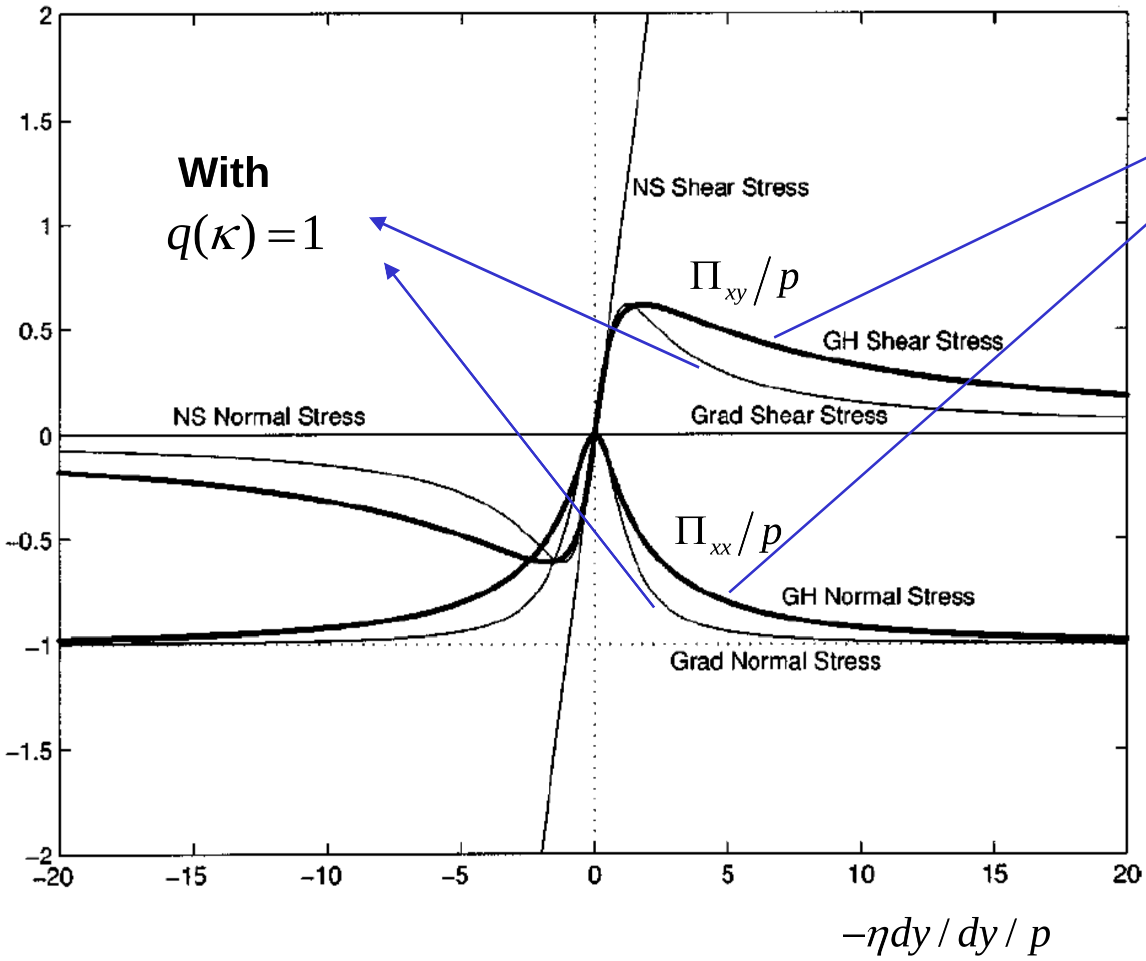
Normalized temperature contours (NSF vs NCCR) in multi-species flowfield around a reentry vehicle (M=23.47, Kn=0.2238; courtesy of J. H. Ahn)

Validation II: 1-d force-driven compressible Poiseuille gas flow (velocity shear dominated; *PoF* 2011)



$$\varepsilon_{h_{\text{wall}}} \equiv \frac{ah}{RT_{\text{wall}}} : \text{ Richardson no., } \mathrm{Kn} \equiv \sqrt{\frac{\pi}{2}} \frac{\eta_{\text{wall}} \sqrt{RT_{\text{wall}}}}{p_{\text{reservoir}} h} : \text{ Knudsen no.}$$

Different role of q(kappa) term in velocity shear



With $q(\kappa)\equiv\frac{\sinh\kappa}{\kappa}$

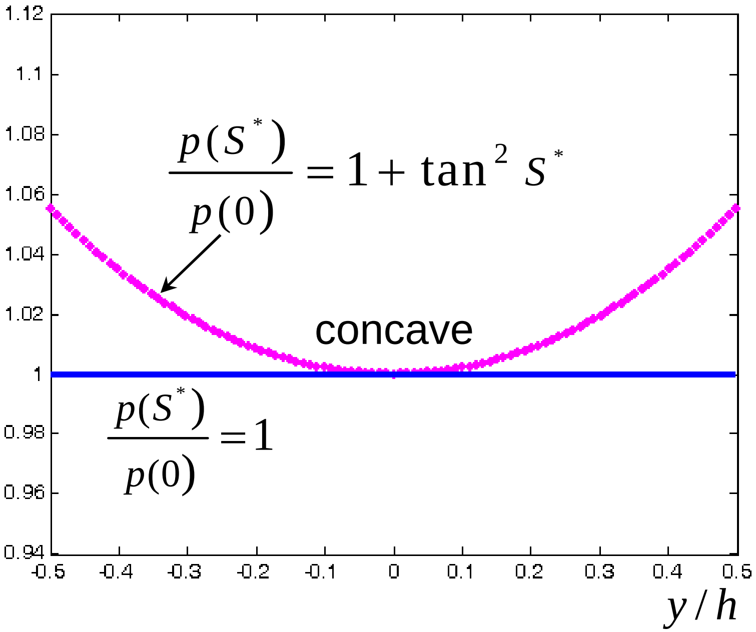
$$\left(\frac{\Pi_{xy}}{p}\right)^2=-\frac{3}{2}\left(1+\frac{\Pi_{yy}}{p}\right)\frac{\Pi_{yy}}{p}$$

Local kinematic

stress constraint

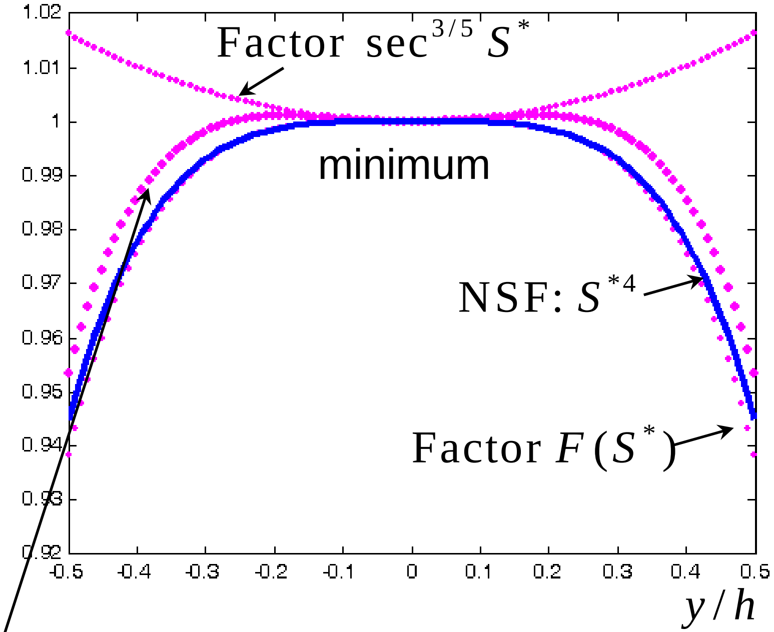
BGK or relaxation is fairly good approximation, in stark contrast with the shock structure case!

Validation II: Fully *analytical* solution (Kn=0.1, $\varepsilon_{h_w} = 0.6$)



Pressure profile across channel

(NSF: blue, NCCR: cyan)



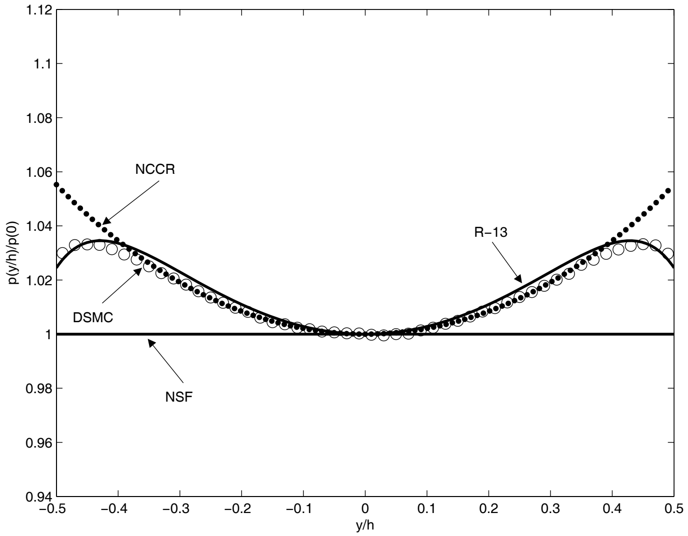
$$\frac{T(S^*)}{T(0)} = \sec^{3/5} S^* \left[1 - \frac{\pi \Pr}{816} \frac{T_w^{*5}}{T^*(0)} \frac{\varepsilon_{h_w}^2}{\mathrm{Kn}^2} \frac{F(S^*)}{S_w^{*4}} \right], \; S^* \equiv \sqrt{\frac{2}{3}} T_w^* \varepsilon_{h_w} s^*$$

$$F\left(S^*\right) \equiv 17 \left[\frac{1}{17 \cos^{17/5} S^*} - \frac{1}{7 \cos^{7/5} S^*} - \left(\frac{1}{17} - \frac{1}{7} \right) \right]$$

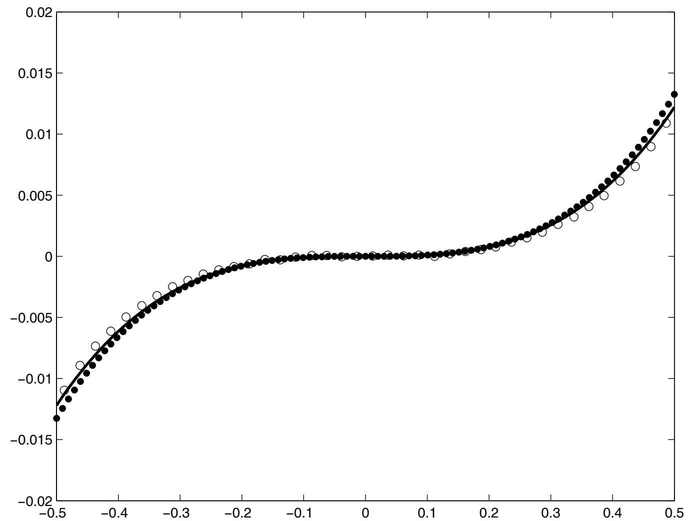
Temperature profile across channel

(NSF: blue, NCCR: cyan)

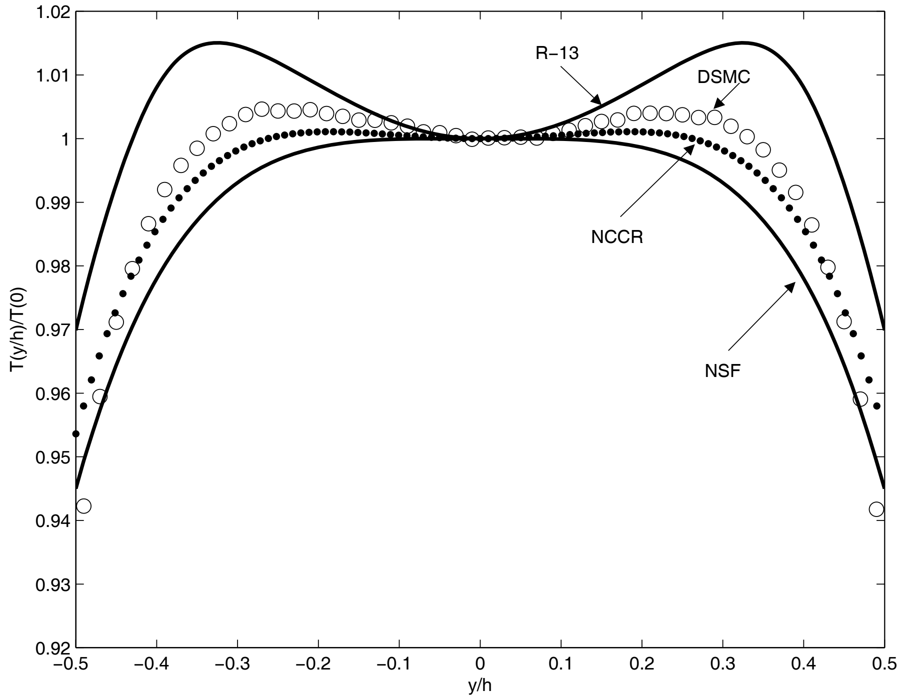
Validation II: comparison with other solutions



Pressure profile across channel



Normal heat flux profile across channel



Temperature profile across channel

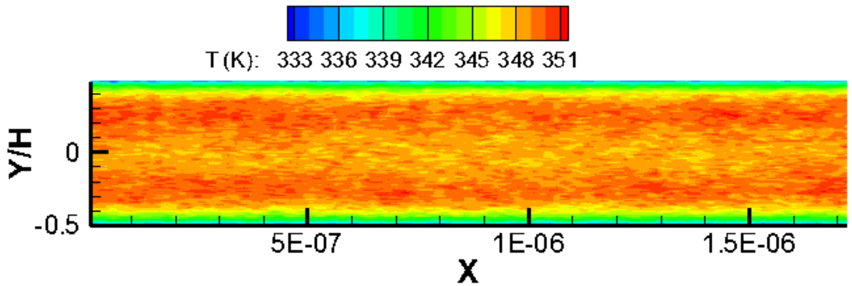
Capturing all the qualitative features predicted by the DSMC

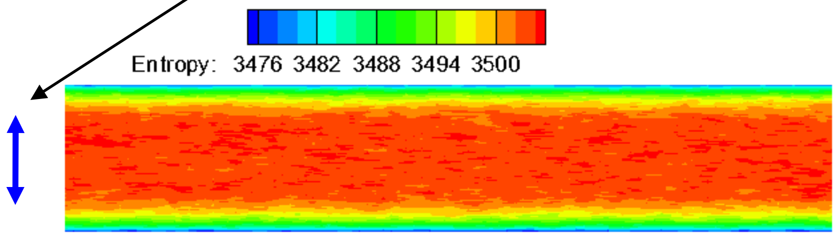
Cf. DSMC (Bird), Regularized-13 (Struchtrup)

An exotic prediction: heat conduction from cold to hot

Non-Fourier behavior in NCCR

$$\hat{Q}_y = \frac{3}{\left(3+2\hat{\Pi}^2_{xy_{NSF}}\right)}\left(\hat{Q}_{y_{NSF}} + \hat{a}\hat{\Pi}_{xy_{NSF}}\right); \; a \text{ is force}$$





Temperature contours

(Courtesy of E. Roohi)

Heat transfer from

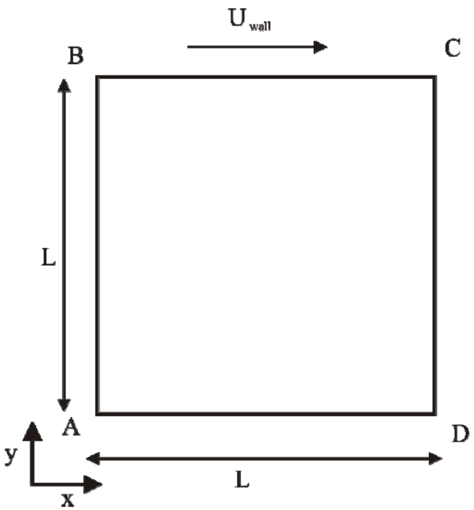
Cold (center) to hot

H function contours

$$\text{Cf. } H(t) = \sum_{bins} \Delta v \frac{N_h(v)}{N} \ln \frac{N_h(v)}{N} \text{ in DSMC}$$

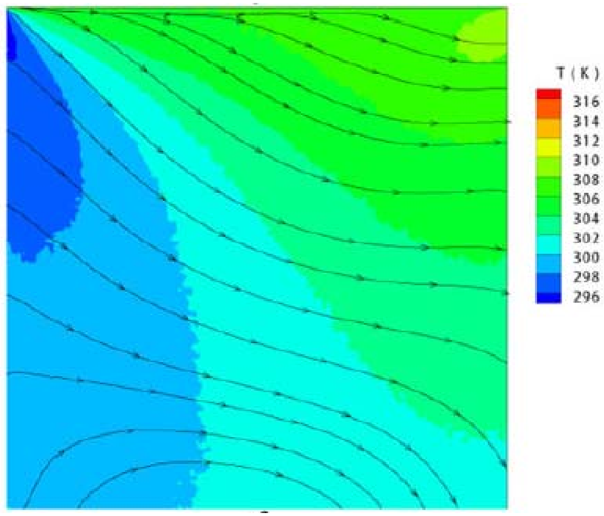
$N_h(v)$: the number of particles in a histogram bin of width Δv

Another example of exotic prediction: lid-driven cavity 2-D gas flows (no force; DSMC; Kn=0.5)

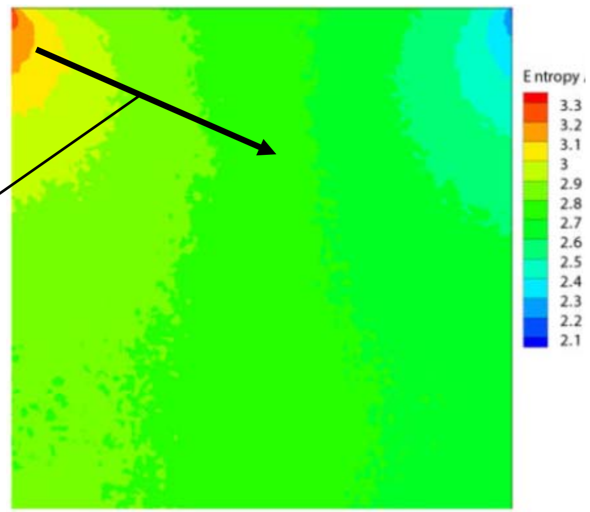


Lid-driven cavity gases

Temperature contours (Courtesy of E. Roohi)



H function contours



Non-Fourier behavior in NCCR

Heat transfer from

Cold (center) to hot

in 2-D

Part III.

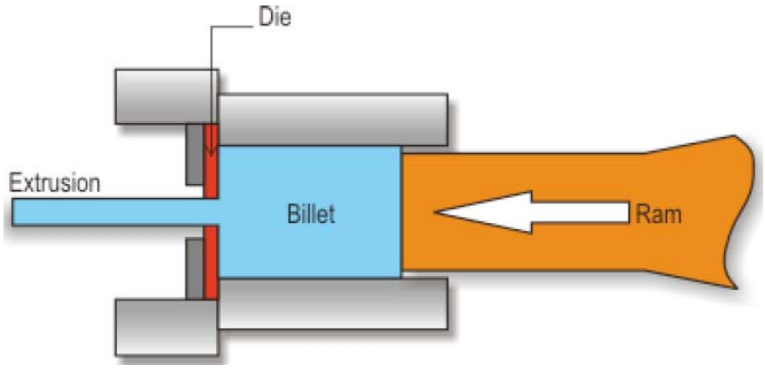
New Potential Applications of the Present Framework

Loss of convergence of viscoelastic flow solutions in complex fluids (polymer solutions; rheology)

Viscoelastic fluids are complex fluids that have "memory" (the state-of-stress depends on the flow history)

Visco: friction, irreversibility, loss of memory

Elastic: recoil, internal energy storage



The high-Weissenberg number problem: A 30 year old mystery. *…All methods, without exception, were found to break down at a frustratingly low value of the Weissenberg number around We=1.*

(R. Fattal, R. Kupferman 2005)

Analogy of viscoelastic flow and rarefied gases

$$\mathrm{We} \equiv \tau / T \sim (l / a)/(L / u) \sim \mathrm{Kn} \cdot M \qquad \text{vs} \qquad N_{\delta} \equiv \Pi / p \sim \mathrm{Kn} \cdot M$$

Upper-convected Maxwell equation derived from the stochastic model of dumbbells

Nonlinear coupled constitutive equation derived from the Boltzmann equation

Key observation: almost complete parallel with the gas dynamic problem -> on-going topic

Occurrence of the singularity in the stationary solution of continuum models of carrier transport in semi-conductors

The mathematic singularity problem in the ballistic regime:

…Concerning the second effect, the spike is enhanced by a smaller mesh size, which is an indication of the occurrence of some sort of "singularity" in the stationary solution.

We remark that such phenomena is not a numerical artifact. Indeed, a similar behavior has been observed with different discretization based on kinetic schemes.

(A. M. Anile et al. 2000)

Concluding remarks

- **Pushing the limits of continuum theory**:
  - algebraic nonlinear coupled constitutive relation (NCCR)
  - persistent attack on the key problems—the shock structure problem
  - delicate balancing in juggling terms
- **Going beyond Navier-Stokes-Fourier:**
  - not easy due to no appearance of 1^{st} order term
  - nothing taken for granted such as the possibility of heat transfer from cold to hot
- **Applicable to other unsolved issues:** defeating the high Weissenberg number problem in computational rheology and the ballistic transport of carriers in semi-conductors.

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